

Double Integrals Over Rectangular Regions

- INTRODUCTION
- DEFINITION OF THE DOUBLE INTEGRAL

from:
BZB

■ INTRODUCTION

We have generalized the concept of differentiation to functions with two or more independent variables. How can we do the same with integration, and how can we interpret the results? Let us first look at the operation of antidifferentiation. We can antidifferentiate a function of two or more variables with respect to one of the variables by treating all the other variables as though they were constants. Thus, this operation is the reverse operation of partial differentiation, just as ordinary antidifferentiation is the reverse operation of ordinary differentiation. We write $\int f(x, y) dx$ to indicate that we are to antidifferentiate $f(x, y)$ with respect to x , holding y fixed; we write $\int f(x, y) dy$ to indicate that we are to antidifferentiate $f(x, y)$ with respect to y , holding x fixed.

Partial Antidifferentiation Evaluate:

$$(A) \int (6xy^2 + 3x^2) dy$$

Treating x as a constant and using the properties of antidifferentiation from Section 11-1, we have

$$\begin{aligned} \int (6xy^2 + 3x^2) dy &= \int 6xy^2 dy + \int 3x^2 dy \\ &= 6x \int y^2 dy + 3x^2 \int dy \\ \text{SOLUTION (A)} \quad &= 6x \left(\frac{y^3}{3} \right) + 3x^2(y) + C(x) \\ &= 2xy^3 + 3x^2y + C(x) \end{aligned}$$

The dy tells us we are looking for the antiderivative of $6xy^2 + 3x^2$ with respect to y only, holding x constant.

Notice that the constant of integration actually can be *any function of x alone*, since, for any such function .

$$\frac{\partial}{\partial y} C(x) = 0$$

Check: We can verify that our answer is correct by using partial differentiation:

$$\begin{aligned} \frac{\partial}{\partial y} [2xy^3 + 3x^2y + C(x)] &= 6xy^2 + 3x^2 + 0 \\ &= 6xy^2 + 3x^2 \end{aligned}$$

Now that we have extended the concept of antidifferentiation to functions with two variables, we also can evaluate definite integrals of the form

$$\int_a^b f(x, y) dx \quad \text{or} \quad \int_c^d f(x, y) dy$$

Evaluating a Partial Antiderivative Evaluate, substituting the limits of integration in y if dy is used and in x if dx is used:

$$\int_0^2 (6xy^2 + 3x^2) dy \qquad \int_0^1 (6xy^2 + 3x^2) dx \qquad \text{Example}$$

$$\int (6xy^2 + 3x^2) dy = 2xy^3 + 3x^2y + C(x)$$

According to properties of the definite integral for a function of one variable, we can use any antiderivative to evaluate the definite integral. Thus, choosing $C(x) = 0$, we have

$$\begin{aligned} \int_0^2 (6xy^2 + 3x^2) dy &= (2xy^3 + 3x^2y) \Big|_{y=0}^{y=2} \\ &= [2x(2)^3 + 3x^2(2)] - [2x(0)^3 + 3x^2(0)] \\ &= 16x + 6x^2 \end{aligned}$$

$$\int (6xy^2 + 3x^2) dx = 3x^2y^2 + x^3 + E(y)$$

Thus, choosing $E(y) = 0$, we have

$$\begin{aligned} \int_0^1 (6xy^2 + 3x^2) dx &= (3x^2y^2 + x^3) \Big|_{x=0}^{x=1} \\ &= [3y^2(1)^2 + (1)^3] - [3y^2(0)^2 + (0)^3] \\ &= 3y^2 + 1 \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^2 \left[\int_0^1 (6xy^2 + 3x^2) dx \right] dy &= \int_0^2 (3y^2 + 1) dy \\ &= (y^3 + y) \Big|_{y=0}^{y=2} \\ &= [(2)^3 + 2] - [(0)^3 + 0] = 10 \end{aligned}$$

iterated
integral

■ DEFINITION OF THE DOUBLE INTEGRAL

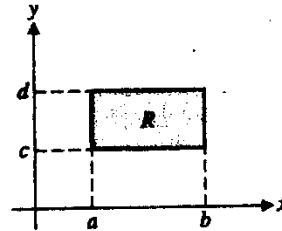
Double Integral

The **double integral** of a function $f(x, y)$ over a rectangle

$$R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$$

is

$$\begin{aligned} \iint_R f(x, y) dA &= \int_a^b \left[\int_c^d f(x, y) dy \right] dx \\ &= \int_c^d \left[\int_a^b f(x, y) dx \right] dy \end{aligned}$$



In the double integral $\iint_R f(x, y) dA$, $f(x, y)$ is called the **integrand** and R is called the **region of integration**. The expression dA indicates that this is an integral over a two-dimensional region. The integrals

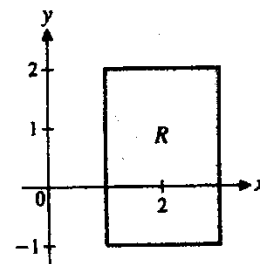
$$\int_a^b \left[\int_c^d f(x, y) dy \right] dx \quad \text{and} \quad \int_c^d \left[\int_a^b f(x, y) dx \right] dy$$

are referred to as **iterated integrals** (the brackets are often omitted), and the order in which dx and dy are written indicates the order of integration. This is not the most general definition of the double integral over a rectangular region; however, it is equivalent to the general definition for all the functions we will consider.

Evaluating a Double Integral Evaluate:

$$\iint_R (x + y) dA$$

over $R = \{(x, y) | 1 \leq x \leq 3, -1 \leq y \leq 2\}$



As a check, we will evaluate the integral both ways:

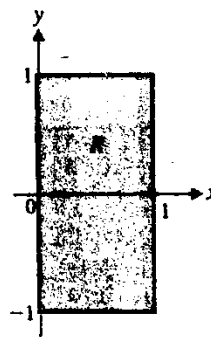
$$\begin{aligned}
 \iint_R (x + y) dA &= \int_1^3 \int_{-1}^2 (x + y) dy dx \\
 &= \int_1^3 \left[\left(xy + \frac{y^2}{2} \right) \Big|_{y=-1}^{y=2} \right] dx \\
 &= \int_1^3 \left[(2x + 2) - \left(-x + \frac{1}{2} \right) \right] dx \\
 &= \int_1^3 \left(3x + \frac{3}{2} \right) dx \\
 &= \left(\frac{3}{2}x^2 + \frac{3}{2}x \right) \Big|_{x=1}^{x=3} \\
 &= \left(\frac{27}{2} + \frac{9}{2} \right) - \left(\frac{3}{2} + \frac{3}{2} \right) = 18 - 3 = 15
 \end{aligned}$$

$$\begin{aligned}
 \iint_R (x + y) dA &= \int_{-1}^2 \int_1^3 (x + y) dx dy \\
 &= \int_{-1}^2 \left[\left(\frac{x^2}{2} + xy \right) \Big|_{x=1}^{x=3} \right] dy \\
 &= \int_{-1}^2 \left[\left(\frac{9}{2} + 3y \right) - \left(\frac{1}{2} + y \right) \right] dy \\
 &= \int_{-1}^2 (4 + 2y) dy \\
 &= (4y + y^2) \Big|_{y=-1}^{y=2} \\
 &= (8 + 4) - (-4 + 1) = 12 - (-3) = 15
 \end{aligned}$$

The Double Integral of an Exponential Function Evaluate:

$$\iint_R 2xe^{x^2+y} dA \quad \text{over} \quad R = \{(x, y) | 0 \leq x \leq 1, -1 \leq y \leq 1\}$$

$$\begin{aligned}
 \iint_R 2xe^{x^2+y} dA &= \int_{-1}^1 \int_0^1 2xe^{x^2+y} dx dy \\
 &= \int_{-1}^1 \left[\left(e^{x^2+y} \right) \Big|_{x=0}^{x=1} \right] dy \\
 &= \int_{-1}^1 (e^{1+y} - e^y) dy \\
 &= (e^{1+y} - e^y) \Big|_{y=-1}^{y=1} \\
 &= (e^2 - e) - (e^0 - e^{-1}) \\
 &= e^2 - e - 1 + e^{-1}
 \end{aligned}$$



Double Integrals Over More General Regions

■ REGULAR REGIONS

Let R be the region graphed in Figure 1. We can describe R with the following inequalities:

$$R = \{(x, y) | x \leq y \leq 6x - x^2, 0 \leq x \leq 5\}$$

The region R can be viewed as a union of vertical line segments. For each x in the interval $[0, 5]$, the line segment from the point $(x, g(x))$ to the point $(x, f(x))$ lies in the region R . Any region that can be covered by vertical line segments in this manner is called a *regular x region*.

Now consider the region S in Figure 2. It can be described with the following inequalities:

$$S = \{(x, y) | y^2 \leq x \leq y + 2, -1 \leq y \leq 2\}$$

The region S can be viewed as a union of horizontal line segments going from the graph of $h(y) = y^2$ to the graph of $k(y) = y + 2$ on the interval $[-1, 2]$. Regions that can be described in this manner are called *regular y regions*.

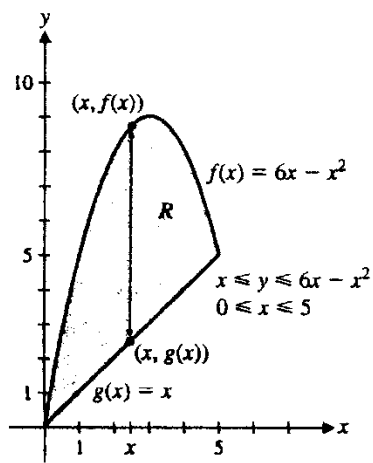


Fig. 1

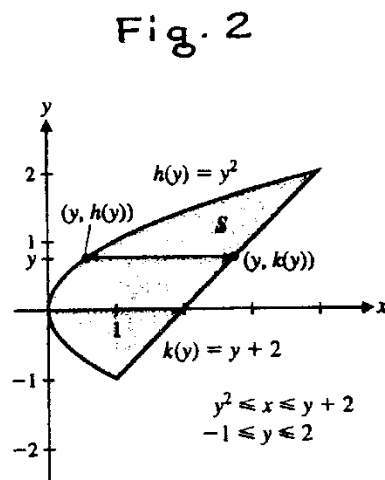


Fig. 2

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In general, *regular regions* are defined as follows:

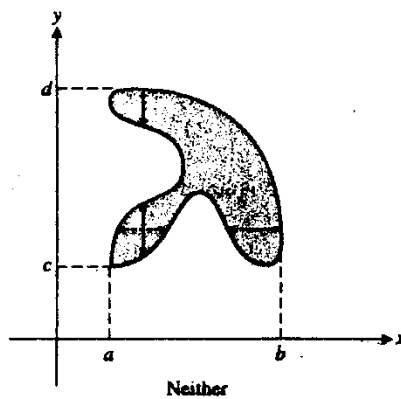
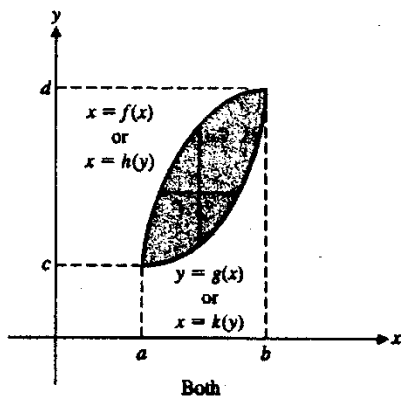
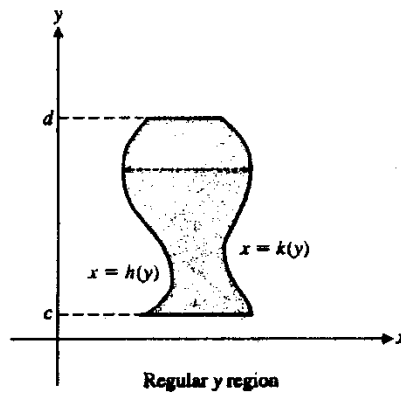
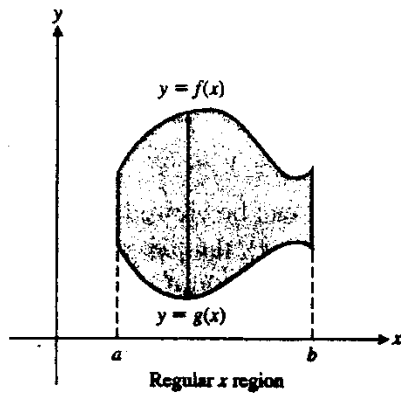
Regular Regions

A region R in the xy plane is a **regular x region** if there exist functions $f(x)$ and $g(x)$ and numbers a and b so that

$$R = \{(x, y) | g(x) \leq y \leq f(x), a \leq x \leq b\}$$

A region R is a **regular y region** if there exist functions $h(y)$ and $k(y)$ and numbers c and d so that

$$R = \{(x, y) | h(y) \leq x \leq k(y), c \leq y \leq d\}$$



Double Integration Over Regular Regions

Regular x Region

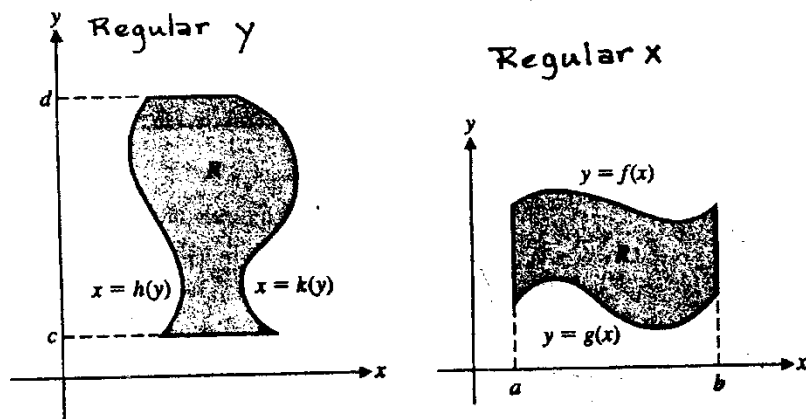
If $R = \{(x, y) | g(x) \leq y \leq f(x), a \leq x \leq b\}$, then

$$\iint_R F(x, y) dA = \int_a^b \left[\int_{g(x)}^{f(x)} F(x, y) dy \right] dx$$

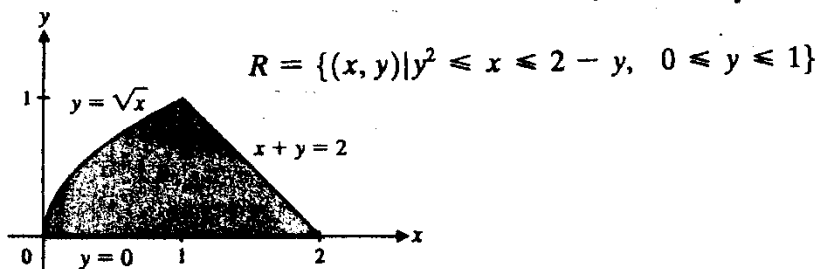
Regular y Region

If $R = \{(x, y) | h(y) \leq x \leq k(y), c \leq y \leq d\}$, then

$$\iint_R F(x, y) dA = \int_c^d \left[\int_{h(y)}^{k(y)} F(x, y) dx \right] dy$$



Evaluating a Double Integral Evaluate $\iint_R (2x + y) dA$, where R is the region bounded by the graphs of $y = \sqrt{x}$, $x + y = 2$, and $y = 0$.



$$\begin{aligned}
\iint_R (2x + y) dA &= \int_0^1 \left[\int_{y^2}^{2-y} (2x + y) dx \right] dy \\
&= \int_0^1 \left[(x^2 + yx) \Big|_{x=y^2}^{x=2-y} \right] dy \\
&= \int_0^1 \{ [(2-y)^2 + y(2-y)] - [(y^2)^2 + y(y^2)] \} dy \\
&= \int_0^1 (4 - 2y - y^3 - y^4) dy \\
&= (4y - y^2 - \frac{1}{4}y^4 - \frac{1}{5}y^5) \Big|_{y=0}^{y=1} \\
&= (4 - 1 - \frac{1}{4} - \frac{1}{5}) - 0 = \frac{51}{20}
\end{aligned}$$

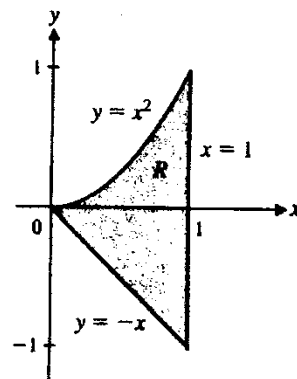
Evaluating a Double Integral Evaluate $\iint_R 2xy dA$, where R is the region bounded by the graphs of $y = -x$ and $y = x^2$, $x \geq 0$, and the graph of $x = 1$.

From the graph we can see that R is a regular x region described by

$$R = \{(x, y) | -x \leq y \leq x^2, 0 \leq x \leq 1\}$$

Thus,

$$\begin{aligned}
\iint_R 2xy dA &= \int_0^1 \left[\int_{-x}^{x^2} 2xy dy \right] dx \\
&= \int_0^1 \left[xy^2 \Big|_{y=-x}^{y=x^2} \right] dx \\
&= \int_0^1 [x(x^2)^2 - x(-x)^2] dx \\
&= \int_0^1 (x^5 - x^3) dx \\
&= \left(\frac{x^6}{6} - \frac{x^4}{4} \right) \Big|_{x=0}^{x=1} \\
&= \left(\frac{1}{6} - \frac{1}{4} \right) - (0 - 0) = -\frac{1}{12}
\end{aligned}$$

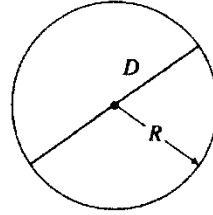


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It is also important to note that the variable limits of integration (when present) are always on the inner integral, and the constant limits of integration are always on the outer integral.

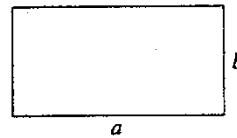
Circle

$$\begin{aligned} R &= \text{Radius} \\ D &= \text{Diameter} \\ D &= 2R \\ A &= \pi R^2 = \frac{1}{4}\pi D^2 && \text{Area} \\ C &= 2\pi R = \pi D && \text{Circumference} \\ \frac{C}{D} &= \pi && \text{For all circles} \\ \pi &\approx 3.141\ 59 \end{aligned}$$



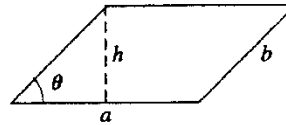
Rectangle

$$\begin{aligned} A &= ab && \text{Area} \\ P &= 2a + 2b && \text{Perimeter} \end{aligned}$$



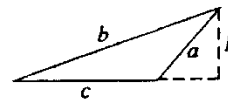
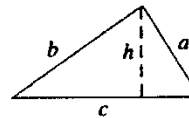
Parallelogram

$$\begin{aligned} h &= \text{Height} \\ A &= ah = ab \sin \theta && \text{Area} \\ P &= 2a + 2b && \text{Perimeter} \end{aligned}$$



Triangle

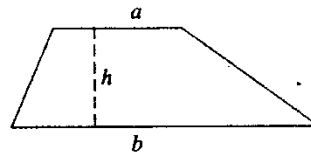
$$\begin{aligned} h &= \text{Height} \\ A &= \frac{1}{2}hc && \text{Area} \\ P &= a + b + c && \text{Perimeter} \\ s &= \frac{1}{2}(a + b + c) && \text{Semiperimeter} \\ A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &&& \text{Area—Heron's formula} \end{aligned}$$



Trapezoid

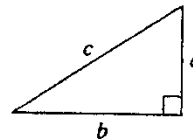
Base a is parallel to base b .

$$\begin{aligned} h &= \text{Height} \\ A &= \frac{1}{2}(a + b)h && \text{Area} \end{aligned}$$



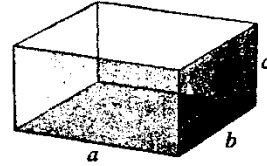
Pythagorean Theorem

$$c^2 = a^2 + b^2$$



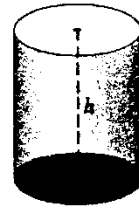
Rectangular Solid

$V = abc$ Volume
 $T = 2ab + 2ac + 2bc$ Total surface area



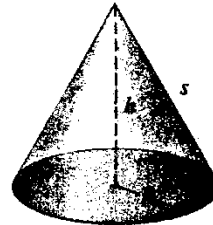
Right Circular Cylinder

$R =$ Radius of base
 $h =$ Height
 $V = \pi R^2 h$ Volume
 $S = 2\pi Rh$ Lateral surface area
 $T = 2\pi R(R + h)$ Total surface area



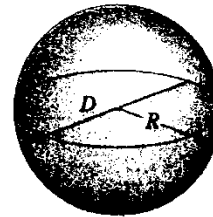
Right Circular Cone

$R =$ Radius of base
 $h =$ Height
 $s =$ Slant height
 $V = \frac{1}{3}\pi R^2 h$ Volume
 $S = \pi Rs = \pi R\sqrt{R^2 + h^2}$ Lateral surface area
 $T = \pi R(R + s) = \pi R(R + \sqrt{R^2 + h^2})$ Total surface area



Sphere

$R =$ Radius
 $D =$ Diameter
 $D = 2R$
 $V = \frac{4}{3}\pi R^3 = \frac{1}{6}\pi D^3$ Volume
 $S = 4\pi R^2 = \pi D^2$ Surface area



Applied Math.
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