3.1. Scalar and vector functions

If all scalars were constant,

\[ x = c , \]  \hspace{1cm} (3.1)

math (and life!) would be pretty dull. Instead, we often deal with functions

\[ x = f(t) . \]  \hspace{1cm} (3.2)

Similarly, constant vectors are dull and not all that useful. We instead consider vector-valued functions that are, in the simplest case, functions of one variable,

\[ \mathbf{v}(t) = f(t) \mathbf{i} + g(t) \mathbf{j} + h(t) \mathbf{k} . \]  \hspace{1cm} (3.3)

We may, for example, consider movement along a curve or trajectory in space,

\[ \mathbf{r}(t) \]

with the corresponding position vector

\[ \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} . \]  \hspace{1cm} (3.4)

Example:

\[ \mathbf{r}(t) = (\cos t) \mathbf{i} + (\sin t) \mathbf{j} + t \mathbf{k} . \]  \hspace{1cm} (3.5)
3.2. Differentiation

Consider a vector $\vec{v}(t)$. The obvious definition of the derivative is

$$\frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} \quad (3.6)$$

$$\frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t}. \quad (3.7)$$

Does this make sense? Here is the geometric picture:

In the limit of small times, we get a tangent vector. In terms of coordinates, if

$$\vec{v}(t) = f(t) \hat{i} + g(t) \hat{j} + h(t) \hat{k}, \quad (3.8)$$

then
\[ \frac{d\vec{v}}{dt} = \frac{df}{dt} \hat{i} + \frac{dg}{dt} \hat{j} + \frac{dh}{dt} \hat{k}. \]  

\[ (3.9) \]

**Warning:** This only works, in general, for Cartesian coordinates.

**Example:**

\[ \vec{v}(t) = e^{3t} \hat{i} + t^2 \hat{j} + \ln(1 + t) \hat{k} \]  

\[ (3.10) \]

\[ \frac{d\vec{v}}{dt} = 3e^{3t} \hat{i} + 2t \hat{j} + \frac{1}{1+t} \hat{k} \]  

\[ (3.11) \]

Here are some basic rules for differentiation:

1. \[ \frac{d}{dt} (\vec{u} + \vec{v}) = \frac{d\vec{u}}{dt} + \frac{d\vec{v}}{dt} \]  

\[ (3.12) \]

2. \[ \frac{d}{dt} [c \vec{v}(t)] = c \frac{d\vec{v}}{dt} \]  

\[ (3.13) \]

3. \[ \frac{d}{dt} [f(t) \vec{v}(t)] = f(t) \frac{d\vec{v}}{dt} + \frac{df}{dt} \vec{v} \]  

\[ (3.14) \]

4. \[ \frac{d}{dt} [\vec{u}(t) \cdot \vec{v}(t)] = \vec{u} \cdot \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \cdot \vec{v} \]  

\[ (3.15) \]

5. \[ \frac{d}{dt} [\vec{u}(t) \times \vec{v}(t)] = \vec{u} \times \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \times \vec{v} \]  

\[ (3.16) \]

**Warning:** The order of the variables is important in the last of these rules.

**Example:**

\[ \frac{d}{dt} (\vec{u} \cdot \vec{v}) = \lim_{\Delta t \to 0} \frac{\vec{u}(t + \Delta t) \cdot \vec{v}(t + \Delta t) - \vec{u}(t) \cdot \vec{v}(t)}{\Delta t} \]  

\[ (3.17) \]

\[ = \lim_{\Delta t \to 0} \frac{(\vec{u} + \Delta \vec{u}) \cdot (\vec{v} + \Delta \vec{v}) - \vec{u} \cdot \vec{v}}{\Delta t} \]  

\[ (3.18) \]
\[
\lim_{\Delta t \to 0} \vec{u} \cdot \frac{\Delta \vec{v}}{\Delta t} + \frac{\Delta \vec{u}}{\Delta t} \cdot \vec{v} + \frac{\Delta \vec{u}}{\Delta t} \cdot \Delta \vec{v} = \vec{u} \cdot \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \cdot \vec{v}
\] (3.19)

Let us now look at some simple examples of differentiating vector-valued functions.

### 3.3. Kinematics in the plane (Cartesian coordinates)

Let

\[\vec{r}(t) = x(t) \hat{i} + y(t) \hat{j}.\] (3.21)

It now follows that

\[\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} = v_x \hat{i} + v_y \hat{j},\] (3.22)

\[\|\vec{v}\| = \sqrt{v_x^2 + v_y^2}; \quad \vec{v} \text{ is tangent to the trajectory},\] (3.23)

\[\vec{a}(t) = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2} \hat{i} + \frac{d^2y}{dt^2} \hat{j} = a_x \hat{i} + a_y \hat{j},\] (3.24)

\[\|\vec{a}\| = \sqrt{a_x^2 + a_y^2}.\] (3.25)

**Example:**

\[\vec{r}(t) = (R \cos \omega t, R \sin \omega t)\] (3.26)

\[\vec{v}(t) = \frac{d\vec{r}}{dt} = (-R\omega \sin \omega t, R\omega \cos \omega t)\] (3.27)

\[\|\vec{v}\| = R\omega = R\dot{\theta}\] (3.28)

\[\vec{a}(t) = (-R\omega^2 \cos \omega t, -R\omega^2 \sin \omega t)\] (3.29)

\[\|\vec{a}\| = R\omega^2 = \text{centripetal acceleration}\] (3.30)

\[\vec{v}(t) \perp \vec{r}(t), \quad \vec{a}(t) \parallel \vec{r}(t)\] (3.31)
Aside:

In our last example, the velocity vector was perpendicular to the position vector. This occurs as long as the position vector is of constant magnitude even if the angular velocity is not constant. Indeed, if we start with

$$\| \vec{r} \| = \text{constant}$$  \hspace{1cm} (3.32)

$$\vec{r} \cdot \vec{r} = \| \vec{r} \|^2 = \text{constant}$$  \hspace{1cm} (3.33)

and we differentiate, we see that

$$\vec{r} \cdot \frac{d\vec{r}}{dt} + \frac{d\vec{r}}{dt} \cdot \vec{r} = 0$$  \hspace{1cm} (3.34)

$$2\vec{r} \cdot \vec{v} = 0$$  \hspace{1cm} (3.35)

$$\vec{r} \perp \vec{v}$$  \hspace{1cm} (3.36)

Up until now, I have described kinematics in the plane using Cartesian coordinates and basis vectors. There is nothing, however, that says that I must limit us to a Cartesian framework.

3.4. Kinematics in the plane (polar coordinates)

We may also use polar coordinates:

$$r \equiv \sqrt{x^2 + y^2}, \quad \theta \equiv \tan^{-1} \frac{y}{x},$$  \hspace{1cm} (3.37)

$$x = r \cos \theta, \quad y = r \sin \theta.$$  \hspace{1cm} (3.38)
The basis vectors $\hat{e}_r$ and $\hat{e}_\theta$ are unit vectors in the $r$ and $\theta$ direction. That is, they are the unit tangent vectors in the direction of increasing $r$ or $\theta$. Let us determine these basis vectors using differentiation.

If we start with
$$\vec{r} = x \vec{i} + y \vec{j} = r \cos \theta \vec{i} + r \sin \theta \vec{j}, \quad (3.39)$$
it follows that
$$\frac{\partial \vec{r}}{\partial r} = \cos \theta \vec{i} + \sin \theta \vec{j} \quad (3.40)$$
and that
$$h_r \equiv \left| \frac{\partial \vec{r}}{\partial r} \right| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1. \quad (3.41)$$
As a result,
$$\hat{e}_r(\theta) = \frac{1}{h_r} \frac{\partial \vec{r}}{\partial r} = \cos \theta \vec{i} + \sin \theta \vec{j}. \quad (3.42)$$

Similarly,
$$\frac{\partial \vec{r}}{\partial \theta} = -r \sin \theta \vec{i} + r \cos \theta \vec{j} \quad (3.43)$$
and
$$h_\theta \equiv \left| \frac{\partial \vec{r}}{\partial \theta} \right| = \sqrt{r^2 \sin^2 \theta + r^2 \cos^2 \theta} = r. \quad (3.44)$$
Thus,
\[ \dot{e}_\theta(\theta) = \frac{1}{h_\theta} \frac{\partial \vec{r}}{\partial \theta} = -\sin \theta \hat{i} + \cos \theta \hat{j}. \] (3.45)

The numbers \( h_r \) and \( h_\theta \) that we used to normalize our basis vectors are **scale factors**. They will play an extremely important role this quarter. Please note that the orientations of \( \hat{e}_r \) and \( \hat{e}_\theta \) depend upon the angle \( \theta \). You should also observe that
\[ \hat{e}_r \cdot \hat{e}_\theta = 0 \Rightarrow \hat{e}_r \perp \hat{e}_\theta \] (3.46)
and that
\[ \frac{d \hat{e}_r}{d \theta} = \hat{e}_\theta, \quad \frac{d \hat{e}_\theta}{d \theta} = -\hat{e}_r. \] (3.47)

To describe the position of an object in polar coordinates, we write
\[ \vec{r} = r \hat{e}_r. \] (3.48)
Please note that \( r(t) \) and \( \theta(t) \) must both be specified since the unit vector \( \hat{e}_r \) depends on \( \theta \). In effect,
\[ \vec{r} = r(t) \hat{e}_r(\theta(t)). \] (3.49)

There are two ways to get the velocity vector. One method is to proceed formally, to note that
\[ \vec{v} = \frac{d \vec{r}}{dt} = \frac{\partial \vec{r}}{\partial r} \frac{dr}{dt} + \frac{\partial \vec{r}}{\partial \theta} \frac{d\theta}{dt}, \] (3.50)
and to now use the relationships between our unit vectors, tangent vectors, and scale factors to write
\[ \frac{d \vec{r}}{dt} = h_r \dot{r} \hat{e}_r + h_\theta \dot{\theta} \hat{e}_\theta \] (3.51)
\[ = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta. \]

The alternative approach is to start with
\[ \vec{r} = r(t) \hat{e}_r(\theta(t)) \] (3.52)
and to differentiate directly, using the product rule. This gives
\[ \vec{v} = \frac{dr}{dt} \hat{e}_r + r \frac{d \hat{e}_r}{dt} \] (3.53)
\[ \vec{v} = \frac{dr}{dt} \hat{e}_r + r \frac{d\hat{e}_r}{d\theta} \frac{d\theta}{dt} \] (3.54)

\[ \vec{v} = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta \] (3.55)

\[ \vec{v} = (\dot{r}, r \dot{\theta}) \] (3.56)

One can now continue on in this way to get the acceleration:

\[ \vec{a} = \ddot{r} \hat{e}_r + \dot{r} \dot{\theta} \hat{e}_\theta + r \ddot{\theta} \hat{e}_\theta - r \dot{\theta}^2 \hat{e}_r \] (3.57)

\[ \vec{a} = (\ddot{r} - r \dot{\theta}^2, 2\dot{r} \dot{\theta} + r \ddot{\theta}) \] (3.58)

where

\[ -r \dot{\theta}^2 \equiv \text{centripetal acceleration} \] (3.59)

\[ 2r \dot{\theta} \equiv \text{Coriolis acceleration} \] (3.60)

**Example:**

\[ r(t) = R, \quad \theta(t) = \omega t \] (3.61)

\[ \vec{v}(t) = (0, R \omega) \] (3.62)

\[ \vec{a}(t) = (-R \omega^2, 0) \] (3.63)

In computing velocity and acceleration we have differentiated the position vector. A closely related operation, computing arc length, includes an extra round of integration.

### 3.5. Arc length

Consider a particle, with position vector \( \vec{r}(t) \), moving along some curve in the plane.
The arc length $s$ of the curve is given by

$$s = \int_{t=a}^{t=b} ds = \int_{t=a}^{t=b} \sqrt{dx^2 + dy^2}$$

(3.65)

$$s = \int_{a}^{b} \sqrt{x'^2 + y'^2} \, dt = \int_{a}^{b} \sqrt{\frac{\dot{r}}{r} \cdot \dot{r}} \, dt$$

(3.66)

$$s = \int_{a}^{b} \left\| \frac{d\vec{r}}{dt} \right\| \, dt$$

(3.67)

The fact that we are integrating a norm means that you should always be integrating a positive quantity. You may need to be careful in how you take your square root!
Example:

\[ x(t) = e^{at} \cos t, \quad y(t) = e^{at} \sin t \]  \hspace{1cm} (3.68)

over the interval \(-\infty \leq t \leq 2\pi\) for \(a > 0\).

This curve was studied by Rene Descartes, Jacob Bernoulli, and others. Jacob Bernoulli so liked the curve that he had it engraved on his tombstone. This curve is called an equiangular spiral because it always cuts radial lines with a constant angle. The logarithmic spiral is surprisingly common. Many of you will immediately recognize this spiral as the cross section of a nautilus shell. A moth that flies into a candle or lantern because it keeps the light at a constant angle also follows this path.

Computing the length of a logarithmic spiral looks difficult, but is remarkably easy. Using Cartesian coordinates,

\[ \vec{r}'(t) = (e^{at} \cos t) \, \vec{i} + (e^{at} \sin t) \, \vec{j} \]  \hspace{1cm} (3.69)

so that

\[ \frac{d\vec{r}}{dt} = e^{at} \left( a \cos t - \sin t, a \sin t + \cos t \right), \]  \hspace{1cm} (3.70)

\[ \|\vec{r}'\| = e^{at} \sqrt{1 + a^2}, \]  \hspace{1cm} (3.71)
and
\[
s = \sqrt{1 + a^2} \int_{-\infty}^{2\pi} e^{at} \, dt = \frac{\sqrt{1 + a^2}}{a} e^{2\pi a} \quad (3.72)
\]

Of course, there was an easier approach. That was to use polar coordinates. You will recall that in polar coordinates we write a planar curve as
\[
\vec{r}(t) = r(t) \hat{r}_r(\theta(t)) \quad (3.73)
\]
and that
\[
\frac{d\vec{r}}{dt} = \frac{\partial \vec{r}}{\partial r} \frac{dr}{dt} + \frac{\partial \vec{r}}{\partial \theta} \frac{d\theta}{dt} \quad (3.74)
\]
\[
= h_r \dot{r} \hat{r}_r + h_\theta \dot{\theta} \hat{e}_\theta
\]
\[
= \dot{r} \hat{r}_r + r \dot{\theta} \hat{e}_\theta.
\]

Since \(\hat{e}_r\) and \(\hat{e}_\theta\) are perpendicular to each other, it now follows that
\[
s = \int \sqrt{\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}} \, dt = \int \sqrt{r^2 + (r\dot{\theta})^2} \, dt \quad (3.75)
\]
\[
= \int \sqrt{dr^2 + r^2 d\theta^2}.
\]

This is the standard formula for arc length in polar coordinates that you first learned in calculus.

Please note that
\[
s \neq \int \sqrt{dr^2 + d\theta^2} \quad (3.76)
\]
That is, a change in the coordinate \(\theta\), \(d\theta\), does not translate directly into a change in arc length. You need the scale factor \(h_\theta = r\) to turn a change in the coordinate \(\theta\) into a change in arc length.
We will see many other scale factors later, when we talk about orthogonal curvilinear coordinates.

For our logarithmic spiral,

$$r(t) = e^{at}, \quad \theta(t) = t$$  \hspace{1cm} (3.77)

so that

$$s = \int \sqrt{r'^2 + (r\theta')^2} \, dt$$  \hspace{1cm} (3.78)

$$= \sqrt{1 + a^2} \int_{-\infty}^{2\pi} e^{at} \, dt$$

$$= \frac{\sqrt{1 + a^2}}{a} e^{2\pi a} .$$

We may extend our discussion of arc length in several useful ways:

1. The formula

$$s = \int_a^b \left\| \frac{d\vec{r}}{dt} \right\| \, dt$$  \hspace{1cm} (3.79)

applies to curves in three dimensions as well as to curves in two dimensions.

2. For many applications, it is helpful to define an arc-length function, $s(t)$, by letting the upper limit of integration be variable,

$$s(t) = \int_a^t \left\| \frac{d\vec{r}}{dt} \right\| \, dt .$$  \hspace{1cm} (3.80)

The rate at which this function changes is your speed,

$$\frac{ds}{dt} = \left\| \frac{d\vec{r}}{dt} \right\| .$$  \hspace{1cm} (3.81)

The representation
\( \vec{r}(s) = x(s) \hat{i} + y(s) \hat{j} + z(s) \hat{k} \), \hspace{1cm} (3.82)

with the arc-length function as a new independent variable, is called the \textit{arc-length parametrization} of the curve. The advantage of this representation is that the unit tangent vector is obtained by simple differentiation,

\[
\vec{T} = \frac{d\vec{r}}{ds} = \left( \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right).
\] (3.83)

Indeed, since

\[
\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt} \frac{dt}{ds} = \frac{d\vec{r}}{dt},
\] (3.84)

it follows that

\[
\| \vec{T} \| = \left\| \frac{d\vec{r}}{ds} \right\| = 1. \] (3.85)

**Example:** For the helix

\( \vec{r}(t) = (a \cos t) \hat{i} + (a \sin t) \hat{j} + b t \hat{k} \), \hspace{1cm} (3.86)

we have

\[
\frac{d\vec{r}}{dt} = (-a \sin t, a \cos t, b)
\] (3.87)

so that

\[
s(t) = \int_0^t \left\| \frac{d\vec{r}}{dt} \right\| dt
\] (3.88)

\[
= \int_0^t \sqrt{a^2 + b^2} \, dt = \sqrt{a^2 + b^2} \, t.
\]

Since

\[
t = \frac{s}{\sqrt{a^2 + b^2}}, \hspace{1cm} (3.89)
\]

it now follows that the arc-length parametrization for our helix is
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just

\[ \vec{r}'(s) = \left( a \cos \frac{s}{\sqrt{a^2 + b^2}}, a \sin \frac{s}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} s \right). \quad (3.90) \]

I will leave it to you to verify that differentiating \( \vec{r}(s) \) gives you a unit tangent vector.