4.1. Scalar fields

A scalar field assigns a scalar, such as temperature or density, to each point in space.

2-d scalar field:

\[ h = f(x, y) \]  \hspace{1cm} (4.1)

3-d scalar field:

\[ h = f(x, y, z) \]  \hspace{1cm} (4.2)

How do we represent such fields? In two dimensions, we may graph the function or we may use contour lines.

Example:

\[ h = 100 - x^2 - y^2 \]  \hspace{1cm} (4.3)

Graph:
For 3-d scalar fields,

\[ h = f(x, y, z), \quad (4.4) \]

we cannot look at the graph in 4 dimensions and so we use level surfaces,

\[ f(x, y, z) = c \quad (4.5) \]

alias

equipotential surfaces
 isolothermal surfaces
 isobaric surfaces

**Example:**

\[ h = 100 - x^2 - y^2 - z^2 \quad (4.6) \]
4.2. Vector fields

In the same way, if we have a vector defined at each point in space, we have a vector field.

2-d vector field:

\[ \vec{v}(x, y) = f(x, y) \hat{i} + g(x, y) \hat{j} \]  
(4.7)

Example:

\[ \vec{F}(x, y) = -\frac{x}{2} \hat{i} - \frac{y}{2} \hat{j} = -\frac{r}{2} \hat{e}_r \]  
(4.8)

\[ \|\vec{F}\| = \frac{r}{2} \]  
(4.9)

3-d vector field:

\[ \vec{v}(x, y, z) = f(x, y, z) \hat{i} + g(x, y, z) \hat{j} + h(x, y, z) \hat{k} \]  
(4.10)

Example:

\[ \vec{g} = \left( -\frac{GM x}{r^3} , -\frac{GM y}{r^3} , -\frac{GM z}{r^3} \right) \]  
(4.11)

where

\[ r = \sqrt{x^2 + y^2 + z^2} \]  
(4.12)
4.3. Directional derivatives

Let's go back to a 2-d scalar field,

\[ h = f(x, y), \quad (4.13) \]

such as

\[ h = 100 - x^2 - 4y^2. \quad (4.14) \]

What is the rate of change of \( h \)? It is easy to answer this question in two special directions:

\[ \frac{\partial h}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \text{rate of change in x direction} \quad (4.15) \]

\[ \frac{\partial h}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \text{rate of change in y direction} \quad (4.16) \]

How about in other directions?

Let's look at the contour lines

\[ f(x, y) = c \quad (4.17) \]

\[ \vec{u} \equiv \text{direction you walk — a unit vector} \quad (4.18) \]

\[ s = \text{distance you walk} \quad (4.19) \]

\[ \vec{r}(s) = x(s) \hat{i} + y(s) \hat{j} \quad (4.20) \]
**Directional derivative in the $\vec{u}$ direction:**

$$D_{\vec{u}} f(\vec{P}) \equiv \lim_{s \to 0} \frac{f(\vec{P} + s \vec{u}) - f(\vec{P})}{s}.$$  \hfill (4.21)

There is also an older notation that suggests how you go about calculating the directional derivative in day-to-day applications:

\[
\frac{df}{ds} = \frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds} \quad \hfill (4.22)
\]

\[
\frac{df}{ds} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{dx}{ds}, \frac{dy}{ds}\right) \quad \hfill (4.23)
\]

\[
\frac{df}{ds} = \nabla f \cdot \vec{u} \quad \hfill (4.24)
\]

**Example:**

$$D_{(1,0)} f = \frac{\partial f}{\partial x} \quad \hfill (4.25)$$

$$D_{(0,1)} f = \frac{\partial f}{\partial y} \quad \hfill (4.26)$$

**Example:**

$$h = 100 - x^2 - 4y^2 \quad \hfill (4.27)$$

$$\vec{P} = (2, 1), \quad \vec{u} = \frac{1}{\sqrt{2}}(1, 1) \quad \hfill (4.28)$$

$$D_{\vec{u}} h(\vec{P}) = \nabla h \cdot \vec{u} \quad \hfill (4.29)$$

$$D_{\vec{u}} h(\vec{P}) = (-2x, -8y)|_{(2,1)} \cdot \frac{1}{\sqrt{2}}(1, 1) \quad \hfill (4.30)$$

$$D_{\vec{u}} h(\vec{P}) = -6\sqrt{2} \quad \hfill (4.31)$$
Example:

\[ f(x, y, z) = 2x^2 + 3y^2 + z^2 \]  

(4.32)

at \( \vec{P} = (2, 1, 3) \)  

(4.33)

in direction \( \hat{i} - 2\hat{k} \)  

(4.34)

\[ \frac{df}{ds} = (4x, 6y, 2z)\big|_{(2, 1, 3)} \cdot \frac{1}{\sqrt{5}} (1, 0, -2) \]  

(4.35)

\[ \frac{df}{ds} = (8, 6, 6) \cdot \frac{1}{\sqrt{5}} (1, 0, -2) \]  

(4.36)

\[ \frac{df}{ds} = -\frac{4}{5} \sqrt{5} \]  

(4.37)

4.4. The gradient

Let’s key in on the gradient of \( f \):

\[ \nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} \]  

in 2d  

(4.38)

\[ \nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \]  

in 3d  

(4.39)

and, in particular, on

\[ \nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}. \]  

(4.40)

This last expression is known variously as

(1) **nabla** — a term introduced by William Hamilton after an Assyrian harplike instrument

(2) **del** — coined by Josiah Willard Gibbs
Nabla follows all of the rules that you might expect from a differentiation operator. For example,

$$\nabla(fg) = \frac{\partial (fg)}{\partial x} \hat{i} + \frac{\partial (fg)}{\partial y} \hat{j} + \frac{\partial (fg)}{\partial z} \hat{k}$$  \hspace{1cm} (4.41)$$

$$\nabla(fg) = f \left( \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) + g \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$  \hspace{1cm} (4.42)$$

$$\nabla(fg) = f \nabla g + g \nabla f$$  \hspace{1cm} (4.43)$$

How about the geometric meaning of $\nabla f$?

If $\vec{u}$ is tangent to a level curve, then

$$\frac{df}{ds} = 0 = \nabla f \cdot \vec{u}$$  \hspace{1cm} (4.44)$$

$$\nabla f \perp \vec{u} \hspace{1cm} \text{or} \hspace{1cm} \nabla f \parallel \vec{n}$$  \hspace{1cm} (4.45)$$

If, in contrast, we choose $\vec{u}$ to be perpendicular to the level curve,

$$\nabla f \cdot \vec{n} = D_{\vec{n}} f$$  \hspace{1cm} (4.46)$$

$$\| \nabla f \| = |D_{\vec{n}} f|$$  \hspace{1cm} (4.47)$$

In summary,

(1) The gradient is perpendicular to level surfaces. It points in the direction of fastest increase.

(2) The magnitude of the gradient is the same as the magnitude of the rate of change in the normal direction.
Find a unit normal to the cone

\[ z^2 = 4(x^2 + y^2) \]  \hspace{2cm} (4.48)

at \( P = (1, 0, 2) \).

**Solution:**

\[ f(x, y, z) = 4(x^2 + y^2) - z^2 = 0 \]  \hspace{2cm} (4.49)

\[ \nabla f = (8x, 8y, -2z) \]  \hspace{2cm} (4.50)

\[ \nabla f(P) = (8, 0, -4) \]  \hspace{2cm} (4.51)

\[ \vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{1}{\sqrt{5}}(2, 0, -1) \]  \hspace{2cm} (4.52)
Example: Polar coordinates revisited

\[ r \equiv \sqrt{x^2 + y^2}, \quad \theta \equiv \tan^{-1} \frac{y}{x}, \quad (4.53) \]

\[ x = r \cos \theta, \quad y = r \sin \theta. \quad (4.54) \]

\[ \vec{r} = x \hat{i} + y \hat{j} \quad (4.55) \]

\[ = r \cos \theta \hat{i} + r \sin \theta \hat{j} \]

You will remember that

\[ \hat{e}_r(\theta) = \frac{1}{h_r} \left( \frac{\partial \vec{r}}{\partial r} \right) = \cos \theta \hat{i} + \sin \theta \hat{j}, \quad (4.56) \]

\[ \hat{e}_\theta(\theta) = \frac{1}{h_\theta} \left( \frac{\partial \vec{r}}{\partial \theta} \right) = -\sin \theta \hat{i} + \cos \theta \hat{j}, \quad (4.57) \]

where

\[ h_r \equiv \left\| \frac{\partial \vec{r}}{\partial r} \right\| = 1, \quad h_\theta \equiv \left\| \frac{\partial \vec{r}}{\partial \theta} \right\| = r. \quad (4.58) \]

There is another method for deriving these unit vectors. Let us look at the \( r \) and \( \theta \) level curves:
(a) Since the gradient is normal to level curves and surfaces, $\nabla r$ should point in the direction of $\hat{e}_r$. In fact,

$$\hat{e}_r = \frac{\nabla r}{\|\nabla r\|} \tag{4.59}$$

Since

$$r = \sqrt{x^2 + y^2}, \tag{4.60}$$

it follows that

$$\nabla r = \left( \frac{x}{r}, \frac{y}{r} \right) = \cos \theta \hat{i} + \sin \theta \hat{j}, \tag{4.61}$$

and that

$$\|\nabla r\| = 1, \tag{4.62}$$

Moreover, since

$$\|\nabla r\| = \frac{dr}{ds} = \frac{1}{h_r} \tag{4.64}$$

in the $r$ direction, we see that

$$h_r = \frac{1}{\|\nabla r\|} = 1. \tag{4.65}$$

(b) The second unit vector is given by
\[ \hat{e}_\theta = \frac{\nabla \theta}{\| \nabla \theta \|} \]  

(4.66)

where

\[ \theta = \tan^{-1} \frac{y}{x}. \]  

(4.67)

In particular,

\[ \nabla \theta = \left( -\frac{y}{r^2}, \frac{x}{r^2} \right) = \frac{1}{r} (-\sin \theta, \cos \theta) \]  

(4.68)

and

\[ \| \nabla \theta \| = \frac{1}{r}. \]  

(4.69)

so that

\[ \hat{e}_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j}. \]  

(4.70)

Moreover, since

\[ \| \nabla \theta \| = \frac{d\theta}{ds} = \frac{1}{h_\theta} \]  

(4.71)

in the theta direction, we see that

\[ h_\theta = \frac{1}{\| \nabla \theta \|} = r. \]  

(4.72)

**Aside:**

I have said that

\[ \left\| \frac{\partial \vec{r}}{\partial \theta} \right\| = h_\theta = r, \]  

(4.73)

but that

\[ \| \nabla \theta \| = \frac{1}{h_\theta} = \frac{1}{r}. \]  

(4.74)

Is it obvious that \( \| \partial \vec{r}/\partial \theta \| \) and \( \| \nabla \theta \| \) should be reciprocals of each other?

Consider

\[ \nabla \theta \cdot \frac{\partial \vec{r}}{\partial \theta} = \left( \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y} \right) \cdot \left( \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta} \right) = \frac{\partial \theta}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial \theta}{\partial y} \frac{\partial y}{\partial \theta}. \]  

(4.75)
We have all the ways that $\theta$ depends on $\theta$, through $x$ and through $y$; the sum equals one. Moreover, since $\nabla \theta$ and $\partial \vec{r} / \partial \theta$ are parallel to each other in polar coordinates,

$$\nabla \theta \cdot \frac{\partial \vec{r}}{\partial \theta} = \| \nabla \theta \| \left\| \frac{\partial \vec{r}}{\partial \theta} \right\| = 1$$

(4.76)

or

$$\left\| \frac{\partial \vec{r}}{\partial \theta} \right\| = \frac{1}{\| \nabla \theta \|}.$$ 

(4.77)

**Bonus:**

One immediate bonus of the above discourse on gradients and scale factors is that it allows us to write the gradient in other coordinate systems. For example, the gradient in polar coordinates is given by

$$\nabla f = (\nabla f \cdot \hat{e}_r) \hat{e}_r + (\nabla f \cdot \hat{e}_\theta) \hat{e}_\theta$$

(4.78)

Moreover, since

$$\nabla f \cdot \hat{e}_r = \frac{df}{ds} \bigg|_{\theta \text{ constant}} = \frac{\partial f}{\partial r}$$

(4.79)

and

$$\nabla f \cdot \hat{e}_\theta = \frac{df}{ds} \bigg|_{r \text{ constant}} = \frac{1}{r} \frac{\partial f}{\partial \theta}$$

(4.80)

the gradient is

$$\nabla f = \frac{\partial f}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{e}_\theta$$

(4.81)

**Example:**

$$f(r, \theta) = r \cos \theta + \tan \theta$$

(4.82)

$$\nabla f = \cos \theta \hat{e}_r + \left( -\sin \theta + \frac{1}{r} \sec^2 \theta \right) \hat{e}_\theta$$

(4.83)