10.1. Cartesian coordinates

We are now interested in volume integrals of the form

\[ I = \int_{V} \int_{V} f(x, y, z) \, dx \, dy \, dz. \]  \hspace{1cm} (10.1)

Volume integrals can and should be handled as iterated integrals.

Example:

Let's compute the weight of a polyhedron, bounded by the planes

\[ x = 0, \quad x = 1, \]  \hspace{1cm} (10.2)

\[ y = 0, \quad y = 2, \]  \hspace{1cm} (10.3)

\[ z = 0, \quad z = 1 + x, \]  \hspace{1cm} (10.4)

that is made of a material of density
\[ \rho(x, y, z) = z. \]

The weight is

\[ W = \int_0^1 \int_0^{1+x} \int_0^1 z \, dz \, dy \, dx \quad (10.5) \]

\[ W = \int_0^1 \int_0^{1+x} \left( \frac{z^2}{2} \right)^{1+x} \, dy \, dx \quad (10.6) \]

\[ W = \int_0^1 \int_0^{1+x} \frac{(1 + x)^2}{2} \, dy \, dx \quad (10.7) \]

\[ W = \int_0^1 \left( \frac{(1 + x)^2 y}{2} \right)^2 \, dy \quad (10.8) \]

\[ W = \int_0^1 (1 + x)^2 \, dx \quad (10.9) \]

\[ W = \frac{(1 + x)^3}{3} \bigg|_0^1 = \frac{7}{3}. \quad (10.10) \]

### 10.2. Orthogonal curvilinear coordinates

For a general system of three curvilinear coordinates, \( u_1, u_2, u_3 \), the corresponding volume integral takes the form

\[ I = \int \int \int_V f(u_1, u_2, u_3) \, dV \quad (10.12) \]

with

\[ dV = \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \left( \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right) \right| \, du_1 \, du_2 \, du_3 \quad (10.13) \]

as the element of volume. In general, one is then left with the
unpleasant task of evaluating the triple-product determinant

\[
\frac{\partial \vec{r}}{\partial u_1} \cdot \left( \frac{\partial \vec{r}}{\partial u_2} \times \frac{\partial \vec{r}}{\partial u_3} \right) = \begin{vmatrix}
\frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\
\frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\
\frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3}
\end{vmatrix} .
\tag{10.14}
\]

If the curvilinear coordinates are orthogonal, the scalar triple-product simplifies,

\[
\frac{\partial \vec{r}}{\partial u_1} \cdot \left( \frac{\partial \vec{r}}{\partial u_2} \times \frac{\partial \vec{r}}{\partial u_3} \right) = h_1 \hat{e}_1 \cdot (h_2 \hat{e}_2 \times h_3 \hat{e}_3) = h_1 h_2 h_3 ,
\tag{10.15}
\]

and our volume integral reduces to

\[
I = \int \int \int_V f(u_1, u_2, u_3) h_1 h_2 h_3 du_1 du_2 du_3 .
\tag{10.16}
\]

**Example: (Spherical)**

Let

\[
f(x, y, z) = z
\tag{10.17}
\]

over the hemisphere

\[
x^2 + y^2 + z^2 \leq 1, \quad z \geq 0.
\tag{10.18}
\]
If \( f(x, y, z) \) is the density of the material, then

\[
W = \int_0^{\pi/2} \int_0^{2\pi} \int_0^r r^2 \sin \theta \cos \theta \, dr \, d\theta \, d\phi
\]  
(10.19)

\[
W = \int_0^{\pi/2} \int_0^{r^4/4} \left( \frac{r^4}{4} \right)^{1/4} \cos \theta \sin \theta \, d\theta \, d\phi
\]  
(10.20)

\[
W = \frac{1}{8} \int_0^{\pi/2} \int_0^\pi \sin 2\theta \, d\theta \, d\phi
\]  
(10.21)

\[
W = -\frac{\pi}{8} \cos 2\theta \bigg|_0^{\pi/2} = \frac{\pi}{4}
\]  
(10.22)

is the weight of the hemisphere.

The corresponding Cartesian integral,

\[
W = \int_{-1}^{+1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^z z \, dz \, dy \, dx
\]  
(10.23)

would be rather more difficult.

Example:

Now for something completely different. Let us go ahead and compute the volume of the hypersphere

\[
x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq R^2
\]  
(10.24)
in 4 dimensions.

I will start by introducing the hyperspherical coordinate system

\[
x_1 = r \sin \theta \sin \psi \cos \phi,
\]  
(10.25)

\[
x_2 = r \sin \theta \sin \psi \sin \phi,
\]  
(10.26)

\[
x_3 = r \sin \theta \cos \psi,
\]  
(10.27)
\[ x_4 = r \cos \theta , \quad (10.28) \]

where \(0 \leq \theta \leq \pi\), \(0 \leq \psi \leq \pi\), and \(0 \leq \phi < 2\pi\). I am sure that there are other formulations that would also work. In general, you will need to make sure that the scale factors are all positive; absolute values may be in order. It is, in any case, easy enough to verify that

\[ x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2 . \quad (10.30) \]

and that the ranges of \(\theta\), \(\phi\), and \(\psi\) cover the right number of orthants.

The position vector is, as you might expect,

\[ \vec{r} = (x_1, x_2, x_3, x_4) \quad (10.31) \]

while the scale factors are simply

\[ h_r = \left| \frac{\partial \vec{r}}{\partial r} \right| = 1 , \quad (10.32) \]

\[ h_\theta = \left| \frac{\partial \vec{r}}{\partial \theta} \right| = r , \quad (10.33) \]

\[ h_\psi = \left| \frac{\partial \vec{r}}{\partial \psi} \right| = r \sin \theta , \quad (10.34) \]

\[ h_\phi = \left| \frac{\partial \vec{r}}{\partial \phi} \right| = r \sin \theta \sin \psi . \quad (10.35) \]

Please note that all of scale factors are nonnegative.

The volume integral is now

\[
V = \int_0^{2\pi} \int_0^\pi \int_0^\pi \int_0^R h_r h_\theta h_\phi h_\psi \, dr \, d\theta \, d\psi \, d\phi \\
= \int_0^{2\pi} \int_0^\pi \int_0^\pi \int_0^R r^3 \sin^2 \theta \sin \psi \, dr \, d\theta \, d\psi \, d\phi \\
= \frac{\pi^2 R^4}{2} . \quad (10.38)
\]
The “surface area” (or rather, an analog of the surface area) for the hypersphere in 4 dimensions can be obtained by simple differentiation,

\[ S = \frac{dV}{dR} = 2\pi^2 R^3. \]  

(10.39)

10.3. Divergence theorem

Earlier, we saw that the curl, a measure of circulation per area, was the basis of Stokes’ theorem, which related surface integrals to line integrals. It stands to reason that the divergence, a measure of net outflux per volume, should, in a similar way, relate volume and surface integrals. This is indeed the case.

**Divergence theorem:**

\[ \int_V \int \nabla \cdot \vec{F} \, dV = \int_{\partial V} \int \vec{F} \cdot d\vec{S} \]  

(10.40)

\[ = \int_{\partial V} \int \vec{F} \cdot \vec{n} \, dS, \]

where \( \vec{n} \) is the outward unit normal.

The vector field \( \vec{F} \) that we’re talking about is assumed to be continuously differentiable on a bounded region.

The divergence theorem is intimately related to our belief that the divergence measures net outflux per unit volume. Indeed, for a sufficiently small element of volume, the divergence theorem implies that

\[ \nabla \cdot \vec{F} \, \Delta V \approx \int_{\partial V} \int \vec{F} \cdot \vec{n} \, dS \]  

(10.41)

so that

\[ \nabla \cdot \vec{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int \int \vec{F} \cdot \vec{n} \, dS. \]

**Example:**

---

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Consider the vector field
\[ \vec{F} = (4x, -2y^2, z^2) \] (10.42)
and the cylinder
\[ x^2 + y^2 \leq 4, \ 0 \leq z \leq 3. \] (10.43)
I’ll let \( S_1 \) be the bottom of the can, \( S_2 \) be the side of the can, and \( S_3 \) be the top of the can.

Let’s evaluate both sides of
\[ \int \int \int_{V} \nabla \cdot \vec{F} \ dV = \int \int \int_{\partial V} \vec{F} \cdot d\vec{S} \] (10.44)
and convince ourselves that we really do get the same answer both ways.

LHS:
If we take the divergence
\[ \nabla \cdot \vec{F} = 4 - 4y + 2z \] (10.45)
and switch to cylindrical coordinates, we obtain.
\[ \nabla \cdot \vec{F} = 4 - 4r \sin \theta + 2z \] (10.46)
The volume integral over this divergence is just
\[
\int \int \int_V (4 - 4r \sin \theta + 2z) \, r \, dz \, d\theta \, dr = 84\pi.
\]

RHS:

We need to do a surface integral over each of \(S_1, S_2,\) and \(S_3\).

\(S_1:\)

\[\vec{n} = -\hat{k} \quad (10.47)\]

\[\vec{F} = (4x, -2y^2, 0) \quad (10.48)\]

\[\int \int_{S_1} \vec{F} \cdot \vec{n} \, dS = 0 \quad (10.49)\]

\(S_2:\)

\[\vec{r} = (2 \cos \theta, 2 \sin \theta, z) \quad (10.50)\]

\[\frac{\partial \vec{r}}{\partial \theta} = (-2 \sin \theta, 2 \cos \theta, 0), \quad \frac{\partial \vec{r}}{\partial z} = (0, 0, 1) \quad (10.51)\]

\[\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z} = (2 \cos \theta, 2 \sin \theta, 0) \quad (10.52)\]

\[\int \int_{S_2} \vec{F} \cdot d\vec{S} = \int \int_{0}^{3\pi} (8 \cos \theta, -8 \sin^2 \theta, z^2) \cdot (2 \cos \theta, 2 \sin \theta, 0) \, d\theta \, dz \quad (10.53)\]

\[\int \int_{S_2} \vec{F} \cdot d\vec{S} = \int \int_{0}^{3\pi} 16(\cos^2 \theta - \sin^3 \theta) \, d\theta \, dz \quad (10.54)\]

\[\int \int_{S_2} \vec{F} \cdot d\vec{S} = 48 \int_{0}^{2\pi} \cos^2 \theta - \sin \theta (1 - \cos^2 \theta) \, d\theta \quad (10.55)\]

\[\int \int_{S_2} \vec{F} \cdot d\vec{S} = \frac{48}{2} \int_{0}^{2\pi} (1 + \cos 2\theta) \, d\theta = 48\pi \quad (10.56)\]
$S_3:\ 
\begin{align*}
   z &= 3, \quad \vec{n} = \vec{k} \\
   \int \int_{S_3} \vec{F} \cdot d\vec{S} &= \int \int 9 \, dS = 36\pi \\
   RHS &= 48\pi + 36\pi = 84\pi
\end{align*}
\tag{10.58}
\tag{10.59}

It’s quite reassuring that we get the same answer both ways.

Example:
Consider the vector field
\[
\vec{F} = (x, y, z)
\tag{10.60}
\]
over the cone
\[
z = \sqrt{x^2 + y^2}, \quad 0 \leq z \leq 1,
\tag{10.61}
\]
with $S_1$ the side and $S_2$ the top.
Let’s evaluate both sides of
\[ \iiint \nabla \cdot \vec{F} \, dV = \iint \vec{F} \cdot \vec{n} \, dS \]  \hspace{1cm} (10.62)
and convince ourselves that we really do get the same answer both ways.

LHS:
\[ \nabla \cdot \vec{F} = 3 \]  \hspace{1cm} (10.63)
\[ 3 \iiint dV = 3 \int_0^{\frac{2\pi}{r}} \int_0^z \int_0^r r \, dr \, d\theta \, dz = \pi \]  \hspace{1cm} (10.64)

RHS:
\[ S_1: \]
\[ \vec{r} = (r \cos \theta, r \sin \theta, r) \]
\[ \frac{\partial \vec{r}}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0), \quad \frac{\partial \vec{r}}{\partial r} = (\cos \theta, \sin \theta, 1) \]  \hspace{1cm} (10.65)
\[ d\vec{S} = \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial r} \, d\theta \, dr = (r \cos \theta, r \sin \theta, -r) \, d\theta \, dr \]  \hspace{1cm} (10.66)
\[ \vec{F} \cdot d\vec{S} = (r \cos \theta, r \sin \theta, r) \cdot (r \cos \theta, r \sin \theta, -r) \, d\theta \, dr \]  \hspace{1cm} (10.67)
\[ S_2: \]
\[ \vec{F} \cdot \vec{n} \, dS = (x, y, z) \cdot \vec{k} \, dS = z \, dS \]  \hspace{1cm} (10.68)
\[ \iint \vec{F} \, d\vec{S} = \iint 1 \, dS = \pi \]  \hspace{1cm} (10.69)

Once again, we get the same answer both ways.

Example: Coulomb’s law

According to Coulomb’s law, the electric field generated by a point charge at the origin,
\[ \vec{E} = \frac{1}{4\pi \varepsilon} \frac{q}{r^2} \hat{e}_r , \quad (10.70) \]

falls of with the square of distance. I am interested in the total flux of this electric field through a closed surface. There are two cases to consider.

1. **Source outside of a volume:**

   In the first case, the point charge is outside the closed surface.

   ![Diagram](image)

   In this case, the divergence theorem implies that
   \[
   \int_S \int \vec{E} \cdot \hat{n} \, dS = \int_V \int \nabla \cdot \vec{E} \, dV \quad (10.71)
   \]
   but
   \[
   \nabla \cdot \vec{E} = \frac{1}{h_r h_\theta h_\phi} \left[ \frac{\partial}{\partial r} (h_\theta h_\phi E_r) + \frac{\partial}{\partial \theta} (h_r h_\phi E_\theta) + \frac{\partial}{\partial \phi} (h_r h_\theta E_\phi) \right] \quad (10.72)
   \]
   \[
   \nabla \cdot \vec{E} = \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left( r^2 \sin \theta \frac{q}{4\pi \varepsilon r^2} \right) = 0 \quad (10.73)
   \]
   so that the total outward flux satisfies
   \[
   \int_S \int \vec{E} \cdot \hat{n} \, dS = 0 . \quad (10.74)
   \]

2. **Source inside of volume:**

   In the second case, the charged particle at the origin is inside a closed surface \( S \).

   ![Diagram](image)
In this case, we can no longer use the divergence theorem in a straightforward way. An important assumption of the divergence theorem is that the vector field be continuously differentiable over the volume of interest. In this case, in contrast, we have a singularity smack dab in the middle of the volume, at the origin. Computing the flux directly by performing a surface integral over $S$ is also problematic since $S$ is some general and mathematically unspecified closed surface. We need to be sneaky.

I’ll begin by constructing a small sphere $V'$ centered at the origin and inside $V$ that surrounds the point charge. Please note that

$$\nabla \cdot \vec{E} = 0 \quad \text{for} \quad V - V', \quad (10.75)$$

that is, for the volume bounded by the two surfaces $S'$ and $S$. Thus, by the divergence theorem,

$$\int \int_S \vec{E} \cdot d\vec{S} + \int \int_{S'} \vec{E} \cdot d\vec{S} = 0 \quad (10.76)$$

so that

$$\int \int_S \vec{E} \cdot d\vec{S} = -\int \int_{S'} \vec{E} \cdot d\vec{S} . \quad (10.77)$$

Note, however, that the surface integral over $S'$ is straightforward, since $S'$ is the surface of a sphere. Indeed,

$$-\int \int_{S'} \vec{E} \cdot d\vec{S} = -\frac{q}{4\pi \epsilon} \int \int \frac{1}{a^2} \hat{e}_r \cdot (-\hat{e}_r) \ dS . \quad (10.78)$$

Moreover, we’ve previously seen that
\[ dS = \left\| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} \right\| d\theta d\phi = a^2 \sin \theta d\theta d\phi \quad (10.79) \]

over the surface of a sphere. It follows that

\[ \iint_S \vec{E} \cdot \vec{n} \, dS = \frac{q}{\varepsilon}, \quad q \in V. \quad (10.80) \]

We thus have two cases:

\[ \iint_S \vec{E} \cdot \vec{n} \, dS = \begin{cases} 
0, & q \notin V \\
\frac{q}{\varepsilon}, & q \in V
\end{cases} \quad (10.81) \]

This is sometimes known as \textbf{Gauss’s law of electrostatics}.

If the charge is distributed over the volume with some density,

\[ q = \iiint_V \rho(x, y, z) \, dV, \quad (10.82) \]

Gauss’s law still applies and

\[ \iint_S \vec{E} \cdot \vec{n} \, dS = \iiint_V \frac{\rho}{\varepsilon} \, dV. \quad (10.83) \]

Sweeping many technical difficulties under the carpet (see potential theory),

\[ \iiint_V \nabla \cdot \vec{E} \, dV = \iiint_V \frac{\rho}{\varepsilon} \, dV. \quad (10.84) \]

Since the volume is completely arbitrary, the integrands must be equal so that

\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}. \quad (10.85) \]

This is one of Maxwell’s equations.

In electrostatics, it is common to think of the electric field as the negative of a potential,

\[ \vec{E} = -\nabla \phi. \quad (10.86) \]

Gauss’s law now yields \textbf{Poisson’s equation},

\[ 10 - 13 \]
\[ \nabla \cdot \nabla \phi = -\frac{\rho}{\varepsilon}, \quad (10.87) \]
a partial differential equation, that in Cartesian coordinates, takes the form
\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -\frac{\rho}{\varepsilon}. \quad (10.88) \]
One typically solves this equation knowing the charge distribution in a certain domain and knowing the potential on the boundary surface of this domain. In the special case where there are no charges inside the domain, Poisson’s equation reduces to Laplace’s equation:
\[ \nabla \cdot \nabla \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad (10.89) \]

**Example:** Chemical Diffusion

Let \( u \) be the concentration of some chemical. The amount of this chemical in some small volume \( V \) changes either because of production or destruction of this chemical, or because of flux through the surface of the volume,
\[ \frac{d}{dt} \int_V \int u \, dV = \int_V \int f \, dV - \int_S \int \vec{J} \cdot \vec{n} \, dS \quad (10.90) \]
Note, however, that the divergence theorem implies that
\[ \int_S \int \vec{J} \cdot \vec{n} \, dS = \int_V \int \nabla \cdot \vec{J} \, dV, \quad (10.91) \]
so that
\[ \int_V \int \left( \frac{\partial u}{\partial t} - f + \nabla \cdot \vec{J} \right) dV = 0. \quad (10.92) \]
We thus have the continuity equation
\[ \frac{\partial u}{\partial t} = f - \nabla \cdot \vec{J}. \quad (10.93) \]
If we assume Fick’s law, so that the flux is proportion to the gradient in concentration,
\[ \vec{J} = -D \nabla u, \quad (10.94) \]
we are lead to
\[
\frac{\partial u}{\partial t} = f(u) + D \nabla \cdot \nabla u \tag{10.95}
\]
or to the reaction-diffusion equation
\[
\frac{\partial u}{\partial t} = f(u) + D \nabla^2 u \tag{10.96}
\]
built around the Laplacian
\[
\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{10.97}
\]

As a special case, consider a steady state with no sources. This produces Laplace’s equation,
\[
\nabla^2 u = 0 \tag{10.98}
\]
10.4. Plausibility argument

Let's look at the 3rd term of the volume integral in the divergence theorem,

\[ \int \int \int_{V} \frac{\partial F_3}{\partial z} \, dz \, dx \, dy = \int \int \left( F_{3}^{top} - F_{3}^{bottom} \right) \, dx \, dy \, . \quad (10.99) \]

At the top,

\[ dx \, dy = \vec{k} \cdot \vec{n} \, dS^{top} \ . \quad (10.100) \]

At the bottom,

\[ dx \, dy = -\vec{k} \cdot \vec{n} \, dS^{bottom} \ . \quad (10.101) \]

Thus,

\[ \int \int \int_{V} \frac{\partial F_3}{\partial z} \, dx \, dy \, dz = \int \int (F_3 \vec{k}) \cdot \vec{n} \, dS \ . \quad (10.102) \]

Similar arguments can be made for the other two terms, yielding:

\[ \int \int \int \nabla \cdot F \, dV = \int \int \vec{F} \cdot \vec{n} \, dS \ . \quad (10.103) \]