11.1. Introduction

We wish to study complex and imaginary numbers. The terminology is standard, but misleading: complex numbers are not complex; imaginary numbers are not imaginary! They are, however, an extension of real numbers.

We wish to solve

\[ x^2 + 1 = 0, \quad (11.1) \]
\[ x^2 = -1. \quad (11.2) \]

This has no real solutions. So we introduce

\[ i \equiv \sqrt{-1} \quad (11.3) \]

as an imaginary number and

\[ z = x + iy \quad (11.4) \]

as a complex number. This turns out to be sufficient to handle all other solutions to simple algebraic equations.

The main topics that we will cover include

1. complex numbers and algebra,
2. complex functions,
3. complex differentiation,
4. complex integration.

Let us start with complex numbers and algebra. As with vectors, the emphasis will eventually be on analysis rather than algebra.

There are three ways to represent complex numbers:

1. algebraically,
2. as points in the complex plane, and
3. in polar form.
Let’s look at each of these individually.

### 11.2. Complex algebra

Let

\[ z = x + iy \]  \hspace{1cm} (11.5)

and

\[ x = \text{Re} \ z, \ y = \text{Im} \ z. \]  \hspace{1cm} (11.6)

This notation dictates all important algebraic operations:

(a) **addition**

\[ (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \]  \hspace{1cm} (11.7)

(b) **multiplication**

\[ (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \]  \hspace{1cm} (11.8)

(c) **division**

\[ \frac{1}{(x + iy)} = \frac{1}{(x + iy)} \frac{(x - iy)}{(x - iy)} = \frac{x - iy}{x^2 + y^2} \]  \hspace{1cm} (11.9)

**Examples:**

(a)

\[ \frac{1}{1 - i} - \frac{1}{1 + i} = \frac{1+i}{2} - \frac{1-i}{2} = i \]  \hspace{1cm} (11.10)

(b)

\[ \frac{1}{(1+i)^4} = \frac{1}{(1+i)^2} \frac{1}{(1+i)^2} = \frac{1}{(2i)^2} = -\frac{1}{4} \]  \hspace{1cm} (11.11)

(c)

\[ \frac{2 + i}{3 - i} = \frac{(2 + i)(3 + i)}{(3 - i)(3 + i)} = \frac{5 + 5i}{10} = \frac{1}{2}(1 + i) \]  \hspace{1cm} (11.12)
Here is some new notation. You will remember that
\[ z = x + iy . \]  
(11.13)

Now, let
\[ \bar{z} = x - iy \]  
(11.14)
be the \textit{complex conjugate} of \( z \).

Here are some simple rules involving \( z \) and \( \bar{z} \):
\[ \text{Re} \ z = \frac{1}{2} (z + \bar{z}) \]  
(11.15)
\[ \text{Im} \ z = \frac{1}{2i} (z - \bar{z}) \]  
(11.16)
\[ \frac{z_1 + \bar{z}_2}{\bar{z}_1 + \bar{z}_2} = \frac{z_1}{\bar{z}_2} \]  
(11.17)
\[ \bar{z}_1 \bar{z}_2 = z_1 \bar{z}_2 \]  
(11.18)

\section*{11.3. Points in the complex plane}

Let
\[ z = (x, y) . \]  
(11.19)

Accordingly, I will define the \textit{modulus} of \( z \) to be the distance from
the origin to \((x, y)\),
\[ |z| = \sqrt{x^2 + y^2} = \sqrt{z \bar{z}} . \]  
(11.20)

\textit{Example}:
\[ |3 + 4i| = 5 \]  
(11.21)
Aside: You should realize that an expression such as
\[ z_1 < z_2 \]  (11.22)
is, in general, meaningless. However,
\[ |z_1| < |z_2| \]  (11.23)
is fine.

Facts:
\[ |z_1z_2| = |z_1||z_2| \]  (11.24)
\[ |z_1 + z_2| \leq |z_1| + |z_2| \]  (11.25)

Proof of (a):
\[ |z_1z_2|^2 = (z_1z_2)\overline{(z_1z_2)} \]  (11.26)
\[ = (z_1 \bar{z}_1)(z_2 \bar{z}_2) \]
\[ = |z_1|^2 |z_2|^2 \]
\[ |z_1z_2| = |z_1| |z_2| \]  (11.27)

Proof of (b):

Triangle Inequality
11.4. Complex numbers in polar form

Let

\[ x = r \cos \theta, \quad y = r \sin \theta \]  \hspace{1cm} (11.28)

and

\[ z = r \cos \theta + i r \sin \theta = r (\cos \theta + i \sin \theta). \]  \hspace{1cm} (11.29)

Clearly,

\[ r = |z| = \text{modulus of } z \]  \hspace{1cm} (11.30)

and

\[ \theta = \arg z \quad \text{(in radians)} \]  \hspace{1cm} (11.31)

where \( \arg z \) or the argument of \( z \) is only defined to within \( 2\pi \).

If \( \theta \in (-\pi, \pi] \), then

\[ \arg z = \text{Arg } z. \]  \hspace{1cm} (11.32)

where Arg \( z \) is known as the \textbf{principal value} of the argument.

More generally,

\[ \arg z = \text{Arg } z + 2 n \pi. \]  \hspace{1cm} (11.33)

Example:

If

\[ z = (1 + i) = (1, i), \]  \hspace{1cm} (11.34)

then

\[ z = \sqrt{2} \left[ \cos \left( \frac{\pi}{4} \right) + i \sin \left( \frac{\pi}{4} \right) \right]. \]  \hspace{1cm} (11.35)

The polar form is particularly useful for multiplication. Let

\[ z_1 = r_1 (\cos \theta_1 + i \sin \theta_1), \]  \hspace{1cm} (11.36)

\[ z_2 = r_2 (\cos \theta_2 + i \sin \theta_2). \]  \hspace{1cm} (11.37)

Now
\[ z_1 z_2 = r_1 r_2 \left[ (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\sin \theta_1 \cos \theta_2 + \sin \theta_2 \cos \theta_1) \right] \]

so that
\[ z_1 z_2 = r_1 r_2 \left[ \cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2) \right]. \]

Clearly,
\[ |z_1 z_2| = |z_1||z_2| \] 
and
\[ \arg z_1 z_2 = \arg z_1 + \arg z_2, \text{ mod } 2\pi. \]

The expression \((\cos \theta + i \sin \theta)\) plays such a key role that I will denote it in a special way. Let
\[ e^{i\theta} \equiv (\cos \theta + i \sin \theta). \]

This is known as Euler's formula. So far, this is strictly notation. Don't confuse notation with truth. Just because I say the sea is boiling hot, doesn't mean that it is. I will justify this formula later.

**Example:**
\[ e^{i\pi} = -1 \]

As an extension of this notation, I will also write
\[ z = r e^{i\theta}, \]
and
\[ \bar{z} = r e^{-i\theta}. \]

Now, we have a new way of doing multiplication:
\[ z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)} \]
with
\[ |z_1 z_2| = |z_1||z_2| \]
and
\[ \arg z_1 z_2 = \arg z_1 + \arg z_2 . \]  

(11.46)

Example:

\[ (-1+i)(1+\sqrt{3}i) = \sqrt{2} e^{i3\pi/4} 2 e^{i\pi/3} = 2 \sqrt{2} e^{i13\pi/12} \]

Powers are also easy:

\[ z^n = (r e^{i\theta})^n = r^n e^{in\theta} . \]  

(11.47)

Example:

\[ (1+i)^8 = (\sqrt{2} e^{i\pi/4})^8 = 16 e^{i2\pi} = 16 \]  

(11.48)

Example:

\[ (\sqrt{3}+i)^9 = (2e^{i\pi/6})^9 = 2^9 e^{i3\pi/2} = 512 (-i) = -512 i \]  

(11.49)

Interestingly, our notation also allows us to write down some very complicated trigonometric identities in a very straightforward manner.

11.4.1. De Moivre’s formula

Let \( r = 1, \) so that

\[ (e^{i\theta})^n = e^{in\theta} . \]  

(11.50)

It would appear that

\[ (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta . \]  

(11.51)

Examples:

For \( n = 2, \)

\[ \cos 2\theta + i \sin 2\theta = (\cos^2 \theta - \sin^2 \theta) + 2 i \cos \theta \sin \theta \]  

(11.52)

so that

\[ \cos 2\theta = \cos^2 \theta - \sin^2 \theta \]  

(11.53)

and

\[ \sin 2\theta = 2 \sin \theta \cos \theta . \]  

(11.54)
For \( n = 3 \),

\[
\cos 3\theta + i \sin 3\theta = \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta
\]

so that

\[
\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta
\]

and

\[
\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta
\]

You can continue on in this way to higher power of \( n \).

### 11.4.2. Roots

Our new notation also facilitates finding roots.

**Example:**

Let us find all numbers \( z \) that satisfy

\[
z^3 = 1 .
\]

That is, we would like all of the third roots of unity. Let’s write each side of our equation in polar form:

\[
(r e^{i\theta})^3 = 1
\]

\[
r^3 e^{3i\theta} = 1 e^{i(0+2k\pi)}
\]

The modulus of the left-hand side must equal the modulus of the right hand side and so

\[
r^3 = 1 \quad \text{or} \quad r = 1 .
\]

In the same way, the argument on the left must equal the argument on the right,

\[
3\theta = 2k\pi ,
\]

so that

\[
\theta = \frac{2k\pi}{3} = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots
\]

It thus follows that
\[ z_1 = e^{0i} = 1, \quad (11.64) \]
\[ z_2 = e^{(2\pi/3)i} = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \quad (11.65) \]
\[ = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \]
and
\[ z_3 = e^{(4\pi/3)i} = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) \quad (11.66) \]
\[ = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \]
are three distinct roots of unity. After this, the roots start repeating.

This same procedure can be extended to higher roots of unity.

**Example:**

If
\[ z^n = (r e^{i\theta})^n = 1, \quad (11.67) \]
so that
\[ r^n e^{in\theta} = 1 e^{i(0 + 2k\pi)}, \quad (11.68) \]
we have
\[ r = 1 \quad \text{and} \quad n\theta = 0 + 2k\pi. \quad (11.69) \]
Thus
\[ \theta = \frac{2\pi k}{n}. \quad (11.70) \]

It is now clear that the \( n \) roots of unity,
\[ e^0, \ e^{(2\pi/n)i}, \ e^{(2\pi/n)i}, \ldots, \ e^{[2(n-1)\pi/n]}, \quad (11.71) \]
are spaced uniformly around the unit circle. Here are pictures for \( n = 2, 3, \) and 4:
We need not limit our discussion roots of unity.

Example:

\[ (-8i)^{1/3} = \left[ 8 e^{i(-\pi/2 + 2n\pi)} \right]^{1/3} = 2 e^{-i\pi/6} e^{2n\pi i/3} \]
\[ = 2 e^{-i\pi/6}, \quad 2 e^{i\pi/2}, \quad 2 e^{7\pi i/6} \] (11.72)

11.5. Complex functions

Complex numbers alone are dull. Complex functions,  
\[ w = f(z), \] (11.73)
that map the complex plane back to itself,  
\[ f: \mathbb{C} \to \mathbb{C}, \] (11.74)
are more interesting. In general, the points in the domain of our function have a real and an imaginary part,  
\[ w = f(x + iy). \] (11.75)
Similarly, the points in the range of function have a real part and an imaginary part,  
\[ w = u(x, y) + iv(x, y). \] (11.76)

Example:
Let  
\[ w = z^2, \] (11.77)
Here,
\[ u = x^2 - y^2 \]  
and
\[ v = 2xy \]. \hfill (11.81)

How do we graph complex functions? They’re not as easy to graph as real function. After all, the domain is of dimension two and range is of dimension two. It stands to reason, that the graph must be of dimension four. Our general strategy will therefore be to think about what the function does to regions, rather than points.

**Example:**

Consider
\[ w = f(z) = z + (2 + 3i) . \] \hfill (11.82)
Example:

Consider

\[ w = f(z) = iz = e^{i\pi/2} r e^{i\theta}, \quad (11.83) \]

\[ w = r e^{i(\theta + \pi/2)}. \quad (11.84) \]

Example:

Consider

\[ w = f(z) = z^2 = r^2 e^{i2\theta}. \quad (11.85) \]