15.1. Taylor Series

Let $C$ be a circle, centered at the origin and let $f(z)$ be analytic inside and on this circle. For $z$ inside this circle, we have, by Cauchy’s integral formula,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(t)}{t-z} \, dt,$$

which we may rewrite as

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(t)}{t} \frac{1}{1-(z/t)} \, dt.$$  \hspace{1cm} (15.2)

Now, let

$$\alpha \equiv \frac{z}{t}. \hspace{1cm} (15.3)$$

Since

$$|\alpha| = \left| \frac{z}{t} \right| = \frac{r}{R} < 1,$$  \hspace{1cm} (15.4)

it follows that

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \ldots.$$  \hspace{1cm} (15.5)

We may thus write
\[ f(z) = \frac{1}{2\pi i} \oint \frac{f(t)}{t} \sum_{n=0}^{\infty} \frac{z^n}{t^n} \, dt. \quad (15.6) \]

If we switch the order of summation and integration (a sloppy step!), we see that

\[ f(z) = \sum_{n=0}^{\infty} \left[ \frac{1}{2\pi i} \oint \frac{f(t)}{t^{n+1}} \, dt \right] z^n \quad (15.7) \]

so that

\[ f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) z^n. \quad (15.8) \]

This is a **Maclaurin series**.

Is the interchange of the infinite sum and the integral legal? Let's be careful. Since

\[ \frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \ldots + \frac{\alpha^N}{1-\alpha}, \quad (15.9) \]

we may write

\[ f(z) = \sum_{n=0}^{N-1} \frac{1}{n!} f^{(n)}(0) z^n + R_N, \quad (15.10) \]

where

\[ R_N = \frac{1}{2\pi i} \oint \frac{f(t)}{t} \frac{(z/t)^N}{1-(z/t)} \, dt. \quad (15.11) \]

Note, however that

\[ |R_N| \leq \frac{1}{2\pi} \frac{M (r/R)^N}{R[1-(r/R)]} 2\pi R \leq M \frac{R}{R-r} \left( \frac{r}{R} \right)^N \quad (15.12) \]

and that this term goes to zero as \( N \to \infty \).

If the circle were centered at \( z_0 \), then

\[ \frac{1}{t-z} = \frac{1}{(t-z_0)-(z-z_0)} \quad (15.13) \]

\[ = \frac{1}{t-z_0} \frac{1}{1-\frac{z-z_0}{t-z}}. \]

We would then get
\[ f(z) = \sum_{n=0}^{\infty} \left[ \frac{1}{2\pi i} \oint \frac{f(t)}{(t-z_0)^{n+1}} \, dt \right] (z-z_0)^n \]  
(15.14)

and the \textit{Taylor series}

\[ f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0) (z-z_0)^n . \]  
(15.15)

Some functions are \textit{entire} and have Taylor series that are valid everywhere. Examples include

\[ e^z = 1 + z + \frac{1}{2} z^2 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} z^n , \]  
(15.16)

\[ \sin z = z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \ldots , \]  
(15.17)

and

\[ \cos z = 1 - \frac{1}{2!} z^2 + \frac{1}{4!} z^4 - \ldots . \]  
(15.18)

Other functions are not entire. When a complex function has a power series that fails to converge, the corresponding real function will also suffer.

\textit{Example:}

Consider

\[ \frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^{2n} . \]  
(15.19)

What is the radius of convergence for this function? Let’s use the ratio test. The ratio test tells us that we can look at

\[ \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \rho . \]  
(15.20)

If \( \rho < 1 \), the series \textit{converges}. If \( \rho > 1 \), the series \textit{diverges}. If \( \rho = 1 \), the ratio test fails to tell us what happens.

In our case,

\[ \rho = \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = x^2 \]  
(15.21)
and the series converges if
\[ x^2 < 1. \] (15.22)

Now, it’s not obvious why the Taylor series breaks down if we view the preceding function as a real function. After all, the function
\[ f(x) = \frac{1}{1 + x^2} \] (15.23)
seems so nice. However, \( f(x) \) inherits all of its problems from
\[ f(z) = \frac{1}{1 + z^2} = 1 - z^2 + z^4 - z^6 + \ldots. \] (15.24)
The function \( f(z) \) is not analytic at \( z = \pm i \). This lack of analyticity restricts the radius of convergence.

15.2. Laurent Series
Suppose that \( f(z) \) has a singularity at \( z_0 = 0 \). Let’s try our usual trick of deforming our contour so as to surround a region of analyticity:

By Cauchy’s integral formula,
\[ f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)}{t-z} \, dt - \frac{1}{2\pi i} \oint_{C_2} \frac{f(t)}{t-z} \, dt = I_1 + I_2 \] (15.25)
in the region of analyticity.

For \( I_1, |z| < |t| \), so that

\[
I_1 = \frac{1}{2\pi i} \oint_{C_1} f(t) \frac{1}{t} \frac{1}{1 - (z/t)} dt = \frac{1}{2\pi i} \oint_{C_1} f(t) \frac{\sum_{n=0}^{\infty} z^n}{t^n} dt
\]

(15.26)

Thus

\[
I_1 = \sum_{n=0}^{\infty} a_n z^n
\]

(15.27)

with

\[
a_n = \frac{1}{2\pi i} \oint_{C_1} f(t) \frac{1}{t^{n+1}} dt .
\]

(15.28)

Similarly, for \( I_2, t < z \), so that

\[
I_2 = -\frac{1}{2\pi i} \oint_{C_2} f(t) \frac{dt}{t-z} = \frac{1}{2\pi i} \oint_{C_2} f(t) \frac{1}{z} \frac{1}{1 - (t/z)} dt ,
\]

(15.29)

\[
I_2 = \frac{1}{2\pi i} \oint_{C_1} f(t) \frac{\sum_{n=0}^{\infty} t^n}{z^n} dt ,
\]

(15.30)

\[
I_2 = \sum_{n=1}^{\infty} \left( \frac{1}{2\pi i} \oint_{C_2} f(t) t^{n-1} dt \right) \frac{1}{z^n} ,
\]

(15.31)

and

\[
I_2 = \sum_{n=-\infty}^{-1} \left( \frac{1}{2\pi i} \oint_{C_2} f(t) t^{n+1} dt \right) z^n .
\]

This may written as

\[
I_2 = \sum_{n=-\infty}^{-1} a_n z^n .
\]

(15.32)

Be sure to notice, though, that the \( a_n \) are now determined by a different contour.

Combining the two series, we get

\[
f(z) = I_1 + I_2 = \sum_{n=-\infty}^{\infty} a_n z^n .
\]

(15.33)
In general, for a singularity \( z_0 \) away from the origin, we get

\[
f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad r_1 < |z - z_0| < r_2.
\]  

(15.34)

\textbf{Example:}

Consider

\[
f(z) = \frac{1}{z - 1}
\]

(15.35)

about \( z = 0 \). There are two cases.

If \( |z| < 1 \), \( f(z) \) is analytic and we get a Taylor series:

\[
\frac{1}{z - 1} = -\frac{1}{1 - z} = -(1 + z + z^2 + z^3 + \ldots)
\]

(15.36)

\[
= -1 - z - z^2 - z^3 - \ldots.
\]

For \( 1 < |z| < \infty \), we expect a Laurent series because of the singularity at \( z = 1 \). We now write

\[
\frac{1}{z - 1} = \frac{1}{z} \cdot \frac{1}{1 - (1/z)} = \frac{1}{z} \left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right).
\]

(15.37)

so that

\[
\frac{1}{z - 1} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots
\]

(15.38)

Let us now consider some more complicated rational functions. If there a several factors in the denominator and they are particularly simple, you may want to deal with product. Usually, however, you are better of using the method of partial fractions to rewrite your rational function as the sum of simpler functions.

\textbf{Example:}

To deal with

\[
f(z) = \frac{-4}{(z + 3)(z - 1)},
\]

(15.39)

I will first use the method of partial fractions to rewrite this
function as
\[
f(z) = \frac{1}{z+3} - \frac{1}{z-1} = \frac{1}{3+z} + \frac{1}{1-z}. \quad (15.40)
\]
This function has singularities at \( z = 1 \) and \( z = -3 \). Let us look at some simple Taylor and Laurent series associated with this function.

About \( z = 0 \), with \( |z| < 1 \), we expect a Taylor series and write
\[
f(z) = \frac{1}{3} \frac{1}{1+\frac{z}{3}} + \frac{1}{1-z} \quad (15.41)
\]
\[
= \frac{1}{3} \left( 1 - \frac{z}{3} + \frac{z^2}{9} + \ldots \right) + (1 + z + z^2 + \ldots)
\]
This reduces to
\[
f(z) = \frac{4}{3} + \frac{8}{9} z + \frac{28}{27} z^2 + \ldots \quad (15.42)
\]

About \( z = 0 \), with \( 1 < |z| < 3 \), we expect a Laurent series and write
\[
f(z) = \frac{1}{3} \frac{1}{\left(1 + \frac{z}{3}\right)} - \frac{1}{z} \frac{1}{\left(1 - \frac{1}{z}\right)} \quad (15.43)
\]
so that
\[
f(z) = \frac{1}{3} \left( 1 - \frac{z}{3} + \frac{z^2}{9} + \ldots \right) - \frac{1}{z} \left( 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots \right) \quad (15.44)
\]
or
\[
f(z) = \ldots - \frac{1}{z^2} - \frac{1}{z} + \frac{1}{3} - \frac{z}{9} + \frac{z^2}{27} + \ldots \quad (15.45)
\]

About \( z = 0 \), with \( |z| > 3 \), we expect a Laurent series and write
\[
f(z) = \frac{1}{z} \frac{1}{\left(1 + \frac{3}{z}\right)} - \frac{1}{z} \frac{1}{\left(1 - \frac{1}{z}\right)} \quad (15.46)
\]

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so that
\[ f(z) = \frac{1}{z} \left( 1 - \frac{3}{z} + \frac{9}{z^2} + \ldots \right) - \frac{1}{z} \left( 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots \right) \] (15.47)

or
\[ f(z) = -\frac{4}{z^2} + \frac{8}{z^3} + \ldots \] (15.48)

Of course, nothing says that the expansion has to be about \( z = 0 \). We could, for example, consider \( 0 < |z - 1| < 4 \). If we let \( t = z - 1 \), then
\[ f(z) = \frac{1}{z+3} - \frac{1}{z-1} = \frac{1}{4+t} - \frac{1}{t} . \] (15.49)

\[ f(t) = \frac{1}{4} \frac{1}{1 + \frac{t}{4}} - \frac{1}{t} = \frac{1}{4} \left( 1 - \frac{t}{4} + \frac{t^2}{16} + \ldots \right) - \frac{1}{t} \] (15.50)

so that
\[ f(z) = -\frac{1}{z-1} + \frac{1}{4} - \frac{(z-1)}{16} + \frac{(z-1)^2}{64} + \ldots \] (15.51)

I won’t give you any really nasty partial fraction expansions, but do keep in mind that partial fraction expansions do sometimes get nasty. Here is an example of what can happen.

**Example:**

The function
\[ f(z) = \frac{1}{(z+1)(z+2)^2} \] (15.52)

has the partial fraction expansion
\[ f(z) = \frac{1}{z+1} - \frac{1}{z+2} - \frac{1}{(z+2)^2} . \] (15.53)

The latter can, in turn, be rewritten
\[ f(z) = \frac{1}{z+1} - \frac{1}{z+2} + \frac{d}{dz} \left( \frac{1}{z+2} \right) . \] (15.54)
This is probably the most useful starting point for obtaining Laurent series for this function.

You can also, of course, use what you know about Taylor series to obtain Laurent series.

**Example:**

\[
f(z) = \frac{\cos z}{z} = \frac{1}{z} \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots \right) = \frac{1}{z} - \frac{z}{2} + \frac{z^3}{4!} + \ldots \quad (15.55)
\]

### 15.3. Singularities

In this section, I will show you how to use Laurent series to classify singularities. Before doing so, let me first remind you of what I mean by a singularity.

**Definition:** A point where the function \( f(z) \) is analytic is a **regular point** of \( f(z) \).

**Definition:** \( z_0 \) is a **singular point** of \( f(z) \) if \( f(z) \) is not analytic at \( z_0 \) but is analytic at some point in every neighborhood of \( z_0 \).

Singularities can be **isolated** or **nonisolated** (unisolated). The function

\[
f(z) = \frac{1}{z} \quad (15.56)
\]

has an isolated singularity at \( z = 0 \). This function is analytic everywhere in the complex plane except at \( z = 0 \), where the function fails to exist. The principal value of the logarithm,

\[
f(z) = \log z \quad (15.57)
\]

is not analytic along its branch cut and thus has nonisolated singularities along its branch cut.

For the moment, I will concentrate on isolated singularities. Suppose, in fact, that we have a function \( f(z) \) that has an isolated singularity at the origin but that otherwise is well-behaved inside
the circle $|z| = R$, 

$$ |z| = R $$

isolated singularity

The Laurent series for this function,

$$ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \quad (15.58) $$

can be broken up into two functions:

$$ f_1(z) = \sum_{n=0}^{\infty} a_n z^n, \quad f_2(z) = \sum_{n=-\infty}^{-1} a_n z^n. \quad (15.59) $$

The function $f_1(z)$ is analytic inside $|z| < R$. The function $f_2(z)$, in contrast, is only analytic for $|z| > 0$. This second function is known as the principal part of $f(z)$; it contains information about the nature of the singularity.

There are three possible scenarios for the principal part:

(a) The principal part is, in fact, absent.

In this case, a finite limit exists as we approach the “singularity”; the function is really analytic. It may, for example, have a removable singularity.

**Example:**

The function

$$ \frac{\sin z}{z} = \frac{1}{z} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots \quad (15.60) $$

has no principal part. The point $z = 0$ is a removable singularity. The function approaches 1 as we approach $z = 0$ and we may as well consider $f(0)$ to be 1.
Example:
The function
\[
\frac{e^z - 1}{z} = \frac{1}{z} \left( 1 + z + \frac{z^2}{2!} + \ldots - 1 \right) = 1 + \frac{z}{2} + \frac{z^2}{3!} + \ldots \quad (15.61)
\]
also has a removable singularity at \( z = 0 \). In each case, we may take \( f(0) = 1 \).

(b) The principal part has a finite number of power-series terms.

We now have a **pole of order** \( k \), where \( k \) is the absolute value of the most negative power in the power series expansion of the principal part. If \( k = 1 \), we have a **simple pole**.

Example:
The function
\[
\frac{e^z}{z} = \frac{1}{z} \left( 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \right) \quad (15.62)
\]
\[
= \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots
\]
has a simple pole at \( z = 0 \).

Example:
The function
\[
\frac{e^z}{z^2} = \frac{1}{z^2} \left( 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \right) \quad (15.63)
\]
\[
= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \ldots
\]
has a second-order pole at \( z = 0 \).

(c) The principal part has an infinite number of power-series terms.
We now have an essential singularity.

Example:

The function
\[ f(z) = e^{1/z} \]  
has the Laurent expansion
\[ f(z) = 1 + \frac{1}{z} + \frac{1}{2! \, z^2} + \frac{1}{3! \, z^3} + \cdots \]  
about \( z = 0 \). It is clear the \( z = 0 \) is an essential singularity. We will see later that essential singularities are especially problematic.

Be sure, by the way, to use the right Laurent series in determining the nature of the principal part and in ascertaining the existence of an essential singularity.

Example:

Consider the power series expansions of
\[ f(z) = \frac{1}{z} \frac{1}{1 - z} \]  
about \( z = 0 \).
(a) For \( 0 < |z| < 1 \),
\[ f(z) = \frac{1}{z} + 1 + z + z^2 + \cdots \]  
(b) For \( |z| > 1 \),
\[ f(z) = -\frac{1}{z^2} - \frac{1}{z^3} + \cdots \]  
Only the first interval (\( 0 < |z| < 1 \)) is relevant at \( z = 0 \). It thus follows that we have a first order or simple pole at the origin.

Example:

The Laurent series
\[ \sum_{n=1}^{\infty} \left( \frac{1}{z} \right)^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \]  

contains infinitely many negative powers of \( z \). Is \( z = 0 \) an essential singularity of the function represented by this series?

The first sum is just

\[ \frac{1}{1 - \frac{1}{z}} - 1 = \frac{1}{z-1} \]  

for \( |z| > 1 \), while the second sum is just

\[ \frac{1}{2} \frac{1}{1 - \frac{z}{2}} = \frac{1}{2 - z} \]

for \( |z| < 2 \). It now pretty clear that we are dealing with the Laurent expansion for the function

\[ f(z) = \frac{1}{z-1} + \frac{1}{2-z} = \frac{1}{(z-1)(2-z)} \]  

over the annulus \( 1 < |z| < 2 \). Since this is not the Laurent expansion in the neighborhood of \( z = 0 \), it does not follow that \( z = 0 \) is an essential singularity of \( f(z) \).

Earlier I said that

\[ \frac{e^z}{z} = \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots \]  

has a simple pole at \( z = 0 \) and that

\[ \frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \cdots \]

has second-order pole at \( z = 0 \). This makes perfect sense since we can eliminate the singularity by multiplying by \( z \), in the first case, and by \( z^2 \) in the second case. This suggests a simple general procedure for determining the order of a pole.

**Fact:** If \( f(z) \) has a pole of order \( n \) at \( z_0 \), then

\[ |(z-z_0)^m f(z)| \to \infty \]

as \( z \to z_0 \) for all \( m < n \), but
has a removable singularity at \( z_0 \).

**Example:**

Using the above procedure, the function

\[
f(z) = \frac{(z - 1)^2}{z(z + 1)^3}
\]

(15.75)

clearly has a simple pole at \( z = 0 \) and a third-order pole at \( z = -1 \).

To take advantage of our “fact,” we will sometimes need to use L’Hospital’s rule.

**L’Hospital’s rule:**

If \( f(z) \) and \( g(z) \) are analytic at \( z_0 \) and \( f(z_0) = g(z_0) = 0 \), but \( g'(z_0) \neq 0 \), then

\[
\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.
\]

(15.76)

**Proof:**

Rewrite

\[
\frac{f(z)}{g(z)} = \frac{f(z) - f(z_0)}{z - z_0} \cdot \frac{z - z_0}{g(z) - g(z_0)}
\]

(15.77)

and take the limit as \( z \) approaches \( z_0 \).

**Example:**

Consider

\[
f(z) = \tan z = \frac{\sin z}{\cos z}
\]

(15.78)

This function has singularities at

\[
z_n = \frac{\pi}{2} + n\pi.
\]

(15.79)

Are these singularities first-order poles? If so, then
should be finite. By L'Hospital's rule, this last limit is equivalent to
\[
\lim_{z \to z_n} \frac{(z - z_n) \cos z + \sin z}{-\sin z} = -1
\] (15.81)
and so we know that we have a simple pole.

**Example:**
The function
\[
f(z) = \frac{1}{e^z - 1}
\] (15.82)
has isolated singularities at
\[
z_n = 2n\pi i.
\] (15.83)
By L'Hospital's rule,
\[
\lim_{z \to z_n} (z - z_n) \frac{1}{e^z - 1} = \lim_{z \to z_n} \frac{1}{e^z}
\] (15.84)
\[
= \frac{1}{e^{2n\pi i}} = 1
\]
This means that all of our singularities are simple poles.

**Example: (Essential Singularity)**
The principal value of
\[
f(z) = e^{-1/z^2} = 1 - \frac{1}{2} + \frac{1}{2!} \frac{1}{2^4} + \ldots
\] (15.85)
has an infinite number of terms. We are dealing with an essential singularity at the origin! This function exhibits wild behavior as we approach the origin (from different directions). If we move in towards the origin along the real axis,
\[
\lim_{x \to 0} e^{-1/x^2} \to 0.
\] (15.86)
If we, in contrast, move in towards the origin along the imaginary axis
\[
\lim_{{y \to 0}} e^{-1/(iy)^2} \to \infty.
\] (15.87)

We will limit ourselves to **meromorphic** functions. Here are two different definitions of meromorphic:

1. A meromorphic function is one whose only singularities are poles.
2. A meromorphic function can be expressed as the ratio of two entire functions.

**Example:**

The function

\[
f(z) = \frac{z^2 + 1}{z^2 - 1}
\]  
(15.88)

is meromorphic and has two simple poles.