Topic 18

Transforms

18.1. Laplace transforms

One of the most useful methods for solving linear, constant-coefficient, homogeneous or nonhomogenous, ordinary differential equations is to use the Laplace transform. The all-important inverse Laplace transform is defined as a complex integral.

Definition: Given a function \( f(t) \), we will write its Laplace transform as

\[
F(s) = \mathcal{L} [f(t)] = \int_{0}^{\infty} e^{-st} f(t) \, dt . \tag{18.1}
\]

Here \( s \) can be either a real variable or a complex quantity.

Example: Let

\[
f(t) = e^{at} . \tag{18.2}
\]

Then

\[
F(s) = \int_{0}^{\infty} e^{-st} e^{at} \, dt = \int_{0}^{\infty} e^{-(s-a)t} \, dt , \tag{18.3}
\]

\[
= \left. \frac{-1}{s-a} e^{-(s-a)t} \right|_{0}^{\infty},
\]

\[
= \frac{1}{s-a} , \quad \text{Re}(s) > a .
\]

The integral transform diverges for \( \text{Re}(s) \leq a \) and converges for \( \text{Re}(s) > a \). Note that equation 18.3 also implies that the inverse transform of \( 1/(s-a) \) is \( e^{at} \).

Example: Let

\[
f(t) = 1 . \tag{18.4}
\]
This is a special case of our previous example with $a = 0$. Thus

$$F(s) = \frac{1}{s}, \quad \text{Re}(s) > 0.$$  \hspace{1cm} (18.5)

**Example:** Consider $\cos \omega t$ and $\sin \omega t$. From our definition,

$$\mathcal{L}[\cos \omega t] = \int_{0}^{\infty} e^{-st} \cos \omega t \, dt \hspace{1cm} (18.6)$$

$$\mathcal{L}[\sin \omega t] = \int_{0}^{\infty} e^{-st} \sin \omega t \, dt . \hspace{1cm} (18.7)$$

Now, observe that

$$\mathcal{L}[e^{i\omega t}] = \int_{0}^{\infty} e^{-st} e^{i\omega t} \, dt = \int_{0}^{\infty} e^{(i\omega-s)t} \, dt \hspace{1cm} (18.8)$$

$$= \frac{1}{s-i\omega} = \frac{s+i\omega}{s^2+\omega^2}, \quad \text{Re} \, s > 0$$

and that

$$\mathcal{L}[e^{-i\omega t}] = \frac{s-i\omega}{s^2+\omega^2}, \quad \text{Re} \, s > 0 \hspace{1cm} (18.9)$$

Equating real and imaginary parts in this equation gives

$$\mathcal{L}[\cos \omega t] = \frac{1}{2} \left( \mathcal{L}[e^{i\omega t}] + \mathcal{L}[e^{-i\omega t}] \right) = \frac{s}{s^2+\omega^2}$$

and

$$\mathcal{L}[\sin \omega t] = \frac{1}{2i} \left( \mathcal{L}[e^{i\omega t}] - \mathcal{L}[e^{-i\omega t}] \right) = \frac{\omega}{s^2+\omega^2}$$

for $\text{Re} \, s > 0$.

Let me point out two important properties of Laplace transforms that make it easier to compute Laplace transforms without performing tedious integrations:
Property 1. If
\[ \mathcal{L}[f(t)] = F(s) , \] (18.10)
then
\[ \mathcal{L}[-tf(t)] = \frac{d}{ds} F(s) . \] (18.11)

Proof: By definition,
\[ F(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt . \] (18.12)
Differentiating both sides of this equation with respect to \( s \) gives
\[ \frac{d}{ds} F(s) = \int_{0}^{\infty} \frac{\partial}{\partial s} (e^{-st}) f(t) \, dt \] (18.13)
\[ = \int_{0}^{\infty} -te^{-st} f(t) \, dt \]
\[ = \mathcal{L}[-tf(t)] . \]

This last property is important because it allows us to replace one (or more) tedious integration by parts with differentiation of a Laplace transform and multiplication by -1.

Example: Compute the Laplace transform of
\[ f(t) = te^t . \] (18.14)
The direct approach would be to evaluate
\[ F(s) = \int_{0}^{\infty} e^{-st} te^t \, dt . \] (18.15)
This requires integration by parts.

The indirect approach is to note that

\[ \mathcal{L} [e^t] = \frac{1}{s-1} \]  \hspace{1cm} (18.16)

so that

\[ \mathcal{L} [t e^t] = -\frac{d}{ds} \frac{1}{s-1} = \frac{1}{(s-1)^2} \]  \hspace{1cm} (18.17)

**Example:** Compute the Laplace transform of

\[ f(t) = t^{13} \]  \hspace{1cm} (18.18)

The direct approach would have us compute

\[ F(s) = \int_0^\infty t^{13} e^{-st} dt, \]  \hspace{1cm} (18.19)

which requires many(!) integrations by part.

Alternatively, one can note that

\[ \mathcal{L} [t^{13}] = (-1)^{13} \frac{d^{13}}{ds^{13}} \mathcal{L} [1] \]  \hspace{1cm} (18.20)

\[ = (-1)^{13} \frac{d^{13}}{ds^{13}} \frac{1}{s} = \frac{(13)!}{s^{14}}. \]

**Property 2.** If

\[ \mathcal{L} [f(t)] = F(s), \]  \hspace{1cm} (18.21)

then

\[ \mathcal{L} [e^{at} f(t)] = F(s - a). \]  \hspace{1cm} (18.22)

**Proof:** By definition,

\[ \mathcal{L} [e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t) dt \]  \hspace{1cm} (18.23)
\[
\begin{align*}
&= \int_0^\infty e^{(a-s)t} f(t) \, dt \\
&= \int_0^\infty e^{-(s-a)t} f(t) \, dt \\
&\equiv F(s-a) .
\end{align*}
\]

Example: Compute the Laplace transform of
\[
f(t) = e^{3t} \sin t .
\]  \hspace{1cm} (18.24)

Earlier, we saw that
\[
\mathcal{L}[\sin t] = \frac{1}{(s^2 + 1)} .
\]  \hspace{1cm} (18.25)

In light of Property 2, it follows that
\[
\mathcal{L}[e^{3t} \sin t] = \frac{1}{(s-3)^2 + 1} .
\]  \hspace{1cm} (18.26)

18.2. Laplace transforms and ODEs

Taking a Laplace transform may be thought of as moving from a \( t \) space, where a problem is difficult to solve, to an \( s \) space, where it is easy. It will allow us to turn ODEs into algebraic equations. In the spring, it will allow you to turn PDEs into ODEs.

![Figure 18.1 Laplace transform pair](image-url)
The real usefulness of the Laplace transform in solving differential equations lies in the fact that the Laplace transform of $\dot{f}(t)$ is very closely related to the Laplace transform of $f(t)$:

$$
L \left[ \frac{df}{dt} \right] = \int_{0}^{\infty} e^{-st} \frac{df}{dt} \, dt .
$$

(18.27)

$$
= e^{-st} f(t) \bigg|_{0}^{\infty} + s \int_{0}^{\infty} e^{-st} f(t) \, dt
$$

$$
= s F(s) - f(0) .
$$

Example:

$$
\frac{dx}{dt} - x = e^{at}
$$

(18.28a)

$$
x(0) = -1
$$

(18.28b)

We begin by Laplace transforming everything in sight:

$$
L \left[ \frac{dx}{dt} \right] - L [x] = L (e^{at}) .
$$

(18.29)

Let’s look at all of the relevant components one by one:

$$
L [x] = \int_{0}^{\infty} e^{-st} x(t) \, dt = X(s)
$$

(18.30)

$$
L \left[ \frac{dx}{dt} \right] = \int_{0}^{\infty} e^{-st} \frac{dx}{dt} \, dt
$$

$$
= s X(s) - x(0)
$$

$$
= s X(s) + 1
$$

$$
L [e^{at}] = \frac{1}{s - a} .
$$

(18.32)

Thus
\[ s X(s) - X(s) + 1 = \frac{1}{s - a} \]  
\[ (s - 1) X(s) = \frac{1}{s - a} - 1 = \frac{1 - s + a}{s - a} \]  
\[ X(s) = \frac{(1 - s + a)}{(s - a)(s - 1)} \]

In transforming our differential equation, we’ve turned it into an algebraic equation that’s easy to solve. Now, however, we have to pay the piper. How do we get back from \( X(s) \) to \( x(t) \)? There are several commonly prescribed methods that one commonly encounters:

(a) Look it up in a table. This is hardly acceptable in this class. If we’re going to look everything up in a table, we may as well start with looking up the answer to our ODE and just drop the whole idea of a Laplace transform.

(b) Partial fractions. This is a favorite technique in many engineering texts and it will certainly do, for many purposes.

Example:

\[ X(s) = \frac{(1 - s + a)}{(s - a)(s - 1)} = \frac{A}{s - a} + \frac{B}{s - 1} \]

\[ = \frac{1}{a - 1} \frac{1}{s - a} - \frac{a}{a - 1} \frac{1}{s - 1} \]

Thus

\[ x(t) = \frac{1}{a - 1} e^{at} - \frac{a}{a - 1} e^t \]

(c) Convolution theorem. This is a useful technique that is, unfortunately, beyond what we have time for.

(d) The obvious method! The obvious method is to make use of an \textit{inverse} Laplace transform. In spite of the fact that this is the obvious approach, it’s not always mentioned, for the simple reason that the inverse Laplace transform involves a complex integral.
The inverse $f(t)$ of the Laplace transform $F(s)$ is given by

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) \, ds,$$  

where the line is parallel to the imaginary $s$-axis and to the right of all of singularities of $F(s)$. The contour is a *Bromwich contour* which is closed off in the left half-plane for $(t > 0)$ and in the right half-plane (yielding $f(t) = 0$) for $t \leq 0$. You can justify the vanishing of the outer arc by, in effect, changing variables so as to rotate the contour and by then using Jordan’s lemma.

**Example:**

$$X(s) = \frac{1 - s + a}{(s - a)(s - 1)}$$  

$$x(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \frac{1 - s + a}{(s - a)(s - 1)} \, ds$$
\[ x(t) = \frac{1}{2\pi i} \oint e^{st} \frac{1-s+a}{(s-a)(s-1)} \, ds \quad (18.41) \]

\[ = 2\pi i \frac{1}{2\pi i} \sum \text{residues of} \left[ \frac{e^{st} (1-s+a)}{(s-a)(s-1)} \right] \]

\[ = \sum \text{residues of} \left[ \frac{e^{st} (1-s+a)}{(s-a)(s-1)} \right] \]

Now, remember, for simple poles,

\[ a_{-1} = \lim_{z \to z_0} \left[ (z-z_0) f(z) \right] \quad (18.42) \]

while for an order-k pole,

\[ a_{-1} = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} \left[ (z-z_0)^k f(z) \right]. \quad (18.43) \]

Thus, for \( s = 1 \),

\[ a_{-1} = \lim_{s \to 1} \frac{e^{st} (1-s+a)}{(s-a)} = \frac{a}{1-a} e^t \quad (18.44) \]

and, for \( s = a \),

\[ a_{-1} = \lim_{s \to a} \frac{e^{st} (1-s+a)}{(s-1)} = \frac{1}{a-1} e^{at}. \quad (18.45) \]

As a result,

\[ x(t) = \frac{1}{(a-1)} \left( e^{at} - a e^t \right). \quad (18.46) \]

Let’s try a problem that’s a little bit harder.

**Example:**

\[ \ddot{x} - 6 \dot{x} + 9 x = t^2 e^{3t}, \quad (18.47a) \]

\[ x(0) = 2, \quad \dot{x}(0) = 6. \quad (18.47b) \]

We observe that

\[ \mathcal{L} [x(t)] = X(s), \quad (18.48) \]
\[ \mathcal{L} [\dot{x}(t)] = s\mathcal{L} [x] - x(0) \quad (18.49) \]
\[ = s X(s) - x(0), \]

\[ \mathcal{L} [\ddot{x}(t)] = s \mathcal{L} [\dot{x}] - \dot{x}(0) \quad (18.50) \]
\[ = s [s X(s) - x(0)] - \dot{x}(0) \]
\[ = s^2 X(s) - s x(0) - \dot{x}(0), \]

and

\[ \mathcal{L} [t^2 e^{3t}] = (-1)^2 \frac{d^2}{ds^2} \frac{1}{s - 3} \quad (18.51) \]
\[ = \frac{2}{(s - 3)^3}. \]

Thus

\[ [s^2 X(s) - 2s - 6] - 6 [s X(s) - 2] + 9 X(s) = \frac{2}{(s - 3)^3}. \quad (18.52) \]

\[ (s^2 - 6s + 9) X(s) = 2 (s - 3) + \frac{2}{(s - 3)^3} \quad (18.53) \]

\[ (s - 3)^2 X(s) = 2 (s - 3) + \frac{2}{(s - 3)^5} \quad (18.54) \]

\[ X(s) = \frac{2}{(s - 3)} + \frac{2}{(s - 3)^5} \quad (18.55) \]

Upon inverting this \( X(s) \), we obtain

\[ x(t) = \frac{1}{2\pi i} \left[ \int_{c-i\infty}^{c+i\infty} \frac{2}{(s - 3)} e^{st} ds + \int_{c-i\infty}^{c+i\infty} \frac{2}{(s - 3)^5} e^{st} ds \right] \quad (18.56) \]

Let’s look at the residues. For the first integral,

\[ a_{-1} = \lim_{s \to 3} (s - 3) \frac{2}{(s - 3)} e^{st} = 2 e^{3t}. \quad (18.57) \]

For the second integral,
\[
\begin{align*}
a_{-1} &= \lim_{s \to 3} \frac{1}{4!} \frac{d^4}{ds^4} \left[ \frac{(s-3)^5 2 e^{st}}{(s-3)^5} \right] \\
&= \frac{1}{12} t^4 e^{3t}.
\end{align*}
\] (18.58)

Thus,

\[
x(t) = 2 e^{3t} + \frac{1}{12} t^4 e^{3t}.
\] (18.59)

**Optional:**

Some Laplace-transform inversions are harder than others.

**Example:** Let

\[
F(s) = \frac{1}{\sqrt{s}}
\] (18.60)

so that

\[
f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{\sqrt{s}} \, ds.
\] (18.61)
One can show that COT, COB, and CI do not contribute anything. That leaves \( T \), for which

\[ s = |s| e^{i\pi} = -x \quad (18.62) \]

\[ \sqrt{s} = \sqrt{|s|} e^{i\pi/2} = i\sqrt{x} \quad (18.63) \]

\[ I_T = \frac{1}{2\pi i} \int_0^\infty e^{-xt} (-dx) \quad (18.64) \]

\[ = -\frac{1}{2\pi} \int_0^{\infty} e^{-xt} \, dx \]

and \( B \), for which,

\[ s = |s| e^{-i\pi} = -x \quad (18.65) \]

\[ \sqrt{s} = \sqrt{|s|} e^{-i\pi/2} = -i\sqrt{x} \quad (18.66) \]

\[ I_B = \frac{1}{2\pi i} \int_0^\infty e^{-xt} (-dx) \quad (18.67) \]

\[ = -\frac{1}{2\pi} \int_0^{\infty} e^{-xt} \, dx . \]

Thus,

\[ I_B = I_T . \quad (18.68) \]

There are no singularities within the closed contour so that the integral around the closed contour is just zero. It quickly follows that

\[ f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \sqrt{s} \, ds = -2I_T \quad (18.69) \]

\[ = \frac{1}{\pi} \int_0^\infty e^{-xt} \, dx . \]

Now, letting

\[ u^2 = xt \ , \quad 2udu = t \, dx , \quad (18.70) \]
it follows that

\[
f(t) = \frac{2}{\pi} \frac{1}{\sqrt{t}} \int_{0}^{\infty} e^{-u^2} du = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{t}} \tag{18.71}
\]

18.3. Fourier transforms

Another integral transform that is extremely useful, for problems on an infinite domain, is the Fourier transform

\[
\mathcal{F}[y(x)] = \int_{-\infty}^{+\infty} y(x) e^{-ikx} \, dx . \tag{18.72}
\]

The corresponding inverse transform is

\[
y(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{F}[y(x)] e^{ikx} \, dk . \tag{18.73}
\]

Example: Suppose we wish to find a solution of

\[
y'' - y = e^{-|x|} \tag{18.74}
\]

that satisfies

\[
\lim_{|x| \to \infty} y(x) = 0 \tag{18.75}
\]

and

\[
\lim_{|x| \to \infty} y'(x) = 0 . \tag{18.76}
\]

Let me start by noting that

\[
\mathcal{F}[y'(x)] = \int_{-\infty}^{+\infty} \frac{dy}{dx} e^{-ikx} \, dx \tag{18.77}
\]

\[
\mathcal{F}[y'(x)] = y(x) e^{-ikx}\bigg|_{-\infty}^{+\infty} + ik \int_{-\infty}^{+\infty} y(x) e^{-ikx} \, dx
\]

\[
\mathcal{F}[y'(x)] = ik \hat{y}(k) .
\]
Likewise,
\[ \mathcal{F}[y''(x)] = -k^2 \hat{y}(k). \] (18.78)

Also,
\[ \mathcal{F}[e^{-|x|}] = \int_{-\infty}^{+\infty} e^{-ikx} e^{-|x|} \, dx \] (18.79)

so that
\[ \mathcal{F}[e^{-|x|}] = \int_{-\infty}^{0} e^{-ikx} e^x \, dx + \int_{0}^{\infty} e^{-ikx} e^{-x} \, dx \] (18.80)

\[ = \int_{0}^{\infty} e^{-(1+ik)x} \, dx + \int_{-\infty}^{0} e^{(1-ik)x} \, dx \]

\[ = \frac{1}{1 + ik} + \frac{1}{1 - ik} = \frac{2}{1 + k^2}. \]

Our differential equation thus reduces to
\[ -k^2 \hat{y} - \hat{y} = \frac{2}{1 + k^2} \] (18.81)

or
\[ \hat{y}(k) = -\frac{2}{(1 + k^2)^2}. \] (18.82)

It now follows that
\[ y(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{-2}{(1 + k^2)^2} e^{ikx} \, dk. \] (18.83)

There are two cases. If \( x > 0 \), we need to close our contour in the upper half plane. If, on the other hand, \( x < 0 \), we need to close our contour in the lower half plane. For \( x > 0 \),
\[ a_{-1} = \lim_{k \to i} \frac{d}{dk} \left[ \frac{(k - i)^2}{(k - i)^2 (k + i)^2} \right] e^{ikx}. \] (18.84)
\[ y(x) = \frac{1}{2\pi} (x + 1) e^{-x}. \] (18.85)

For \( x < 0 \),

\[ a_{-1} = \lim_{k \to -i} \frac{d}{dk} \left[ \frac{-2 e^{ikx}}{(k + i)^2} \right] \] (18.86)

\[ = \lim_{k \to -i} \frac{d}{dk} \left[ \frac{-2 e^{ikx}}{(k - i)^2} \right] \]

\[ = \lim_{k \to -i} (-2) \left[ \frac{(k - i)^2 ixe^{ikx} - e^{ikx} 2(k - i)}{(k - i)^4} \right] \]

\[ = i \frac{1}{2} (x - 1) e^{x}. \]

Thus (note the extra minus because we are now going clockwise),

\[ y(x) = \frac{1}{2\pi} (-2\pi i) a_{-1} = \frac{1}{2} (x - 1) e^{x}. \] (18.87)

The solution may thus be written

\[ y(x) = -\frac{1}{2} \left( 1 + |x| \right) e^{-|x|}. \] (18.88)