Sample Solutions for Assignment 2.

Reading: Chs. 2-3.

1. Exercise 4 of Chapter 2. Please note that there is a typo in the formula in part c: An exponent of $1/4$ is missing. It should say

$$\frac{dR}{dt} = - \left( \frac{a}{4M^{1/4}} \right) R.$$

(a) We are given a differential equation for $N_c(t)$, the number of cells at time $t$, and wish to express it as a differential equation for $m(t)$, the total body mass at time $t$, where $m(t) = m_cN_c(t)$ and $m_c$ is a constant equal to the mass of a single cell. We can write

$$\frac{dm}{dt} = \frac{d}{dt}(m_cN_c(t)) = m_c \frac{dN_c}{dt} = \frac{m_c}{E_c}(Y(t) - Y_cN_c(t)).$$

Replacing $N_c(t)$ on the right-hand side by $m(t)/m_c$, we obtain an equation for $dm/dt$ as a function of $m(t)$ and $Y(t)$:

$$\frac{dm}{dt} = \frac{m_c}{E_c}Y(t) - \frac{Y_c}{E_c}m(t),$$

and substituting the given expression for $Y(t)$,

$$\frac{dm}{dt} = am^{3/4} - bm,$$

where $a = Y_0m_c/E_c$ and $b = Y_c/E_c$.

(b) When $dm/dt = 0$, $am^{3/4} = bm$, so solving for $m$ we find that the mass of the matured organism $M$ is:

$$M = (a/b)^{4}.$$

Returning to the differential equation (1), and factoring out $am^{3/4}$, we can write $dm/dt = am^{3/4}[1 - (b/a)m^{1/4}]$ and replacing $b/a$ by $1/M^{1/4}$ gives the desired form

$$\frac{dm}{dt} = am^{3/4}[1 - (m/M)^{1/4}].$$
(c) Letting \( r = (m/M)^{1/4} \) and \( R = 1 - r \), we can write

\[
\frac{dr}{dt} = \frac{1}{4} \left( \frac{m}{M} \right)^{-3/4} \frac{1}{M} \frac{dm}{dt},
\]

and substituting expression (2) for \( \frac{dm}{dt} \),

\[
\frac{dr}{dt} = \frac{1}{4} \left( \frac{m}{M} \right)^{-3/4} \frac{1}{M} \cdot \frac{a M^{3/4}}{M} \left[ 1 - (m/M)^{1/4} \right] = \frac{a}{4M^{1/4}} \left[ 1 - (m/M)^{1/4} \right].
\]

Finally, using the expression for \( R(t) \),

\[
\frac{dR}{dt} = -\left( \frac{a}{4M^{1/4}} \right) R.
\]

Solving for \( R(t) \), we find

\[
R(t) = \exp \left( -\frac{at}{4M^{1/4}} \right) R(0).
\]

Dividing each side by \( R(0) \) and taking logs on both sides gives

\[
\ln \left( \frac{R(t)}{R(0)} \right) = -\frac{at}{4M^{1/4}},
\]

which shows that if \( \ln(R(t)/R(0)) \) is plotted vs. \( at/(4M^{1/4}) \) it will be a straight line through the origin with slope \(-1\). Note that this holds for any organism regardless of its size.

(d) In (c), we showed that

\[
\frac{dR}{dt} = -\left( \frac{a}{4M^{1/4}} \right) R,
\]

which is an equation describing the rate at which different organisms grow to maturity in terms of their mass \( M \) at maturity. If we introduce a new time variable \( \tau = at/(4M^{1/4}) \), which is adjusted for the mass of the organism, then, using the chain rule,

\[
\frac{dR}{d\tau} = \frac{dR}{dt} \cdot \frac{dt}{d\tau} = \frac{dR}{dt} \cdot \frac{4M^{1/4}}{a},
\]

and equation (3) becomes

\[
\frac{dR}{d\tau} = -R(\tau).
\]

This holds for all animals regardless of their mass \( M \).

Since the rate at which any mammal matures is the same in terms of \( \tau \) and \( \tau \propto M^{-1/4} \), time scales associated with the animal, such as time to maturity or lifetime should scale as \( M^{1/4} \). Since the lifetime of an animal is the product of the number of heartbeats (1.5 billion) and the interval between heartbeats, the interval between heartbeats should also scale as \( M^{1/4} \).
2. Exercise 7 of Chapter 2.

(a) We are told that the number of nodes that currently have $k$ links and gain one when the new node comes into the network is $kp_k(n)/2$. Similarly, the number of nodes that currently have $k-1$ links and gain one when the new node comes into the network is $(k-1)p_{k-1}(n)/2$. Let $N_k(n+1)$ denote the number of nodes with $k$ links after the new node enters the network and $N_k(n)$ denote the number of nodes with $k$ links before the new node enters. Then

$$N_k(n+1) - N_k(n) = \frac{(k-1)p_{k-1}(n)}{2} - \frac{kp_k(n)}{2}.$$  

(b) For a large network, we expect the ratios $p_k(n) \equiv N_k(n)/n$ to be almost constant independent of $n$, so let’s denote these simply as $p_k$. Then the equation in part (a) can be written as

$$(n+1)p_k - np_k = \frac{(k-1)p_{k-1}}{2} - \frac{kp_k}{2};$$

that is,

$$p_k \left(1 + \frac{k}{2}\right) = p_{k-1} \left(\frac{k-1}{2}\right) \Rightarrow \frac{p_k}{p_{k-1}} = \frac{k-1}{k+2}.$$  

[Note that this makes sense only for $k \geq 2$.]

(c)

$$p_k = \left(\frac{k-1}{k+2}\right)p_{k-1} = \left(\frac{k-1}{k+2}\right)\left(\frac{k-2}{k+1}\right)p_{k-2} =$$

$$\left(\frac{k-1}{k+2}\right)\left(\frac{k-2}{k+1}\right)\left(\frac{k-3}{k}\right)p_{k-3} = \ldots = \left(\frac{(k-1)!}{(k+2)(k+1)(k+\cdots+5)}\right)p_2$$

Most of the factors in the numerator and denominator cancel and we are left with

$$p_k = \left(\frac{24}{(k+2)(k+1)k}\right)p_2;$$

that is, for $k$ large, $p_k \propto k^{-3}$.

3. Exercise 4 of Chapter 3.

This should have said your choices are to take 25 annual payments of $400,000 each or to take $5 million dollars up front. [Now, there might be reasons independent of interest rates that would make you want the money in one form or the other. For instance, if you owe a loan shark $4 million dollars and he is threatening to break your legs if you don’t pay immediately, you should take the $5 million dollars up front, for sure! One person mentioned tax differences, which I hadn’t actually thought about.]
The $5$ million lump sum would surely be taxed at a higher rate than the $400,000$ installments. I will just discuss a decision based on interest rates. Suppose you take the $5$ million up front and you deposit it at interest rate $r$ per year, compounded continuously, but at the start of each year you withdraw $x$ to buy cool things. So initially you deposit $5 \times 10^6 - x$. At the end of the first year it has grown to $e^r(5 \times 10^6 - x)$. You then withdraw $x$ more dollars and you are left with

$$e^r(5 \times 10^6 - x) - x = e^r(5 \times 10^6) - x[1 + e^r].$$

At the end of the second year you have

$$e^{2r}(5 \times 10^6) - x[e^r + e^{2r}]$$

and after withdrawing your spending money for the coming year this becomes

$$e^{2r}(5 \times 10^6) - x[1 + e^r + e^{2r}]$$

After 24 years (which is when the last installment payment would be made), you will have

$$e^{24r}(5 \times 10^6) - x[1 + e^r + \ldots + e^{24r}] = e^{24r}(5 \times 10^6) - x \left( \frac{e^{25r} - 1}{e^r - 1} \right). \quad (4)$$

Now suppose you take $400,000$ per year for 25 years. Assume that you earn the same interest rate and have the same spending habits as above (so we are assuming that the amount $x$ that you spend each year is less than or equal to $400,000$). You initially receive $400,000$ and you set $x$ aside for spending. You deposit the remaining $4 \times 10^5 - x$ dollars at interest rate $r$ per year and at the end of one year you have

$$e^r(4 \times 10^5 - x).$$

You now receive your next instalment of $400,000$ and you set aside another $x$ for spending. You add the rest to your savings so that you have

$$[e^r + 1](4 \times 10^5 - x).$$

This earns interest over the next year so that after 2 years you have

$$[e^{2r} + e^r](4 \times 10^5 - x).$$

You then receive your next installment and deposit all but $x$ of that so that you have

$$[e^{2r} + e^r + 1](4 \times 10^5 - x).$$

This continues until you receive your last installment, at which time you have

$$[e^{24r} + \ldots + e^r + 1](4 \times 10^5 - x) = \left( \frac{e^{25r} - 1}{e^r - 1} \right)(4 \times 10^5 - x). \quad (5)$$
By setting the right-hand side of (4) equal to that of (5) and solving for \( r \), we can determine the breakeven point \( r^* \). For \( r < r^* \), it is better to take the installments, while for \( r > r^* \) it is better to take the lumpsum. Another idea is to plot the right-hand sides of (4) and (5) vs. interest rate \( r \) (say, for \( x = 0 \)) and look at where the plots intersect to determine the breakeven point.

![Graph showing savings after 24 years vs. interest rate]

For this problem there was no one right answer. But whatever answer you got, you needed to say what the assumptions were that led to that answer. Were you assuming interest compounded continuously, annually, or at some other rate?

4. In the 1999 movie *Office Space*, a character creates a program that takes fractions of cents that are truncated in a bank’s transactions and deposits them to his own account. This is not a new idea, and hackers who have actually attempted it have been arrested. In this exercise we will simulate the program to determine how long it would take to become a millionaire this way.

Assume that we have access to 50,000 bank accounts. Initially we can take the account balances to be uniformly distributed between, say, $100 and $100,000. The annual interest rate on the accounts is 5%, and interest is compounded daily and added to the accounts, except that fractions of a cent are truncated. These will be deposited to an illegal account that initially has balance $0.

(a) Take the following partial MATLAB code and fill in the indicated lines to simulate the *Office Space* scenario. [Alternatively, you may write a code in a language of your choice to do the same thing.]

```matlab
% Simulate "Office Space" scenario, where fractions of a
```
% penny from legal accounts are transferred to an illegal
% account. Determine how long before illegal account has
% a million dollars.

accounts = 100 + (100000-100)*rand(50000,1); % Sets up 50,000 accounts with
% balances between $100 and $100000.
accounts = floor(100*accounts)/100; % Deletes fractions of a cent from
% initial balances.
illegal = 0; % illegal acct is initially 0.
days = 0;

while illegal < 10^6, % Continue until illegal acct
% has a million dollars
    days = days + 1;
    accounts_new = % Add daily interest to accounts.
        % Fill in formula for value of accounts after interest is added.
    accounts = floor(100*accounts_new)/100; % Delete fractions of a cent.

    illegal = % Add daily interest to illegal acct.
        % Fill in formula for value of illegal acct after interest is added.
    illegal = illegal + % Also add fractions of a cent
        % deleted from other accounts.
        % Fill in an expression for the total amount of money deleted from the other
        % accounts that will now be added to illegal acct. (You may want to use
        % the Matlab 'sum' command.)
end;
days, illegal % Print results.

The first lines of the code set up the initial accounts. The MATLAB function rand
generates uniformly distributed random numbers between 0 and 1. The function
floor takes the largest integer that is less than or equal to its argument. Thus, if
an account value is, say 527.125 dollars, we multiply by 100 to get 52712.5 cents,
then take floor of this number to get 52712 cents. Finally to get the truncated
amount back into dollars, we divide by 100 and have 527.12 dollars.

After setting up the accounts and initializing the illegal account to 0, we will
now iterate until the amount in the illegal account reaches a million dollars (10^6),
using a while loop. Inside that loop, the first thing you need to fill in is a formula
for the value of the accounts after interest is added. (This is stored in the vector
accounts_new). The line after that uses floor to delete fractions of a cent, just
as was done in the initial setup. Next you need to fill in a formula for the value of
the illegal account after interest is added. We will assume that this account can
hold fractional amounts, so it will not be truncated. Finally, you need to add to
the illegal account all of the money that was removed from the other accounts. (You may find the MATLAB `sum` command useful for this.)

Turn in a listing of your code or, at least, the lines that you filled in above as well as the answer to the question of how long does it take to become a millionaire this way.

I filled in the following lines:

```matlab
accounts_new = accounts*(1 + .05/365); % Add daily interest to accounts.
accounts = floor(100*accounts_new)/100; % Delete fractions of a cent.

illegal = illegal*(1 + .05/365); % Add daily interest to illegal acct.
illegal = illegal + sum(accounts_new-accounts); % Also add fractions of a cent deleted from other accounts.
```

I found that it took 3189 days or about 8.7 years to become a millionaire this way.

(b) Without running your code, answer the following questions: On average about how much money would you expect to be added to the illegal account each day due to the embezzlement? Suppose you had access to 100,000 accounts, each initially with a balance of, say $5000. About how much money would be added to the illegal account each day in this case? Explain your answers.

On average, the illegal account would get about 0.005 dollars (half a cent) from each account that it accesses independent of the amount in that account, assuming only that the fractions of a cent in the accounts are uniformly distributed. Hence with access to 100,000 accounts, I would expect to embezzle about 500 dollars a day.