Assignment 2.

Due Friday, Oct. 14.

Reading: Finish Ch 1. Read Secs 2.1-2.3 and Sec. 3.1.

1. Do ONE of the following two problems:

(a) Use MATLAB or another programming language to fit a polynomial of degree 12 to the Runge function

\[ f(x) = \frac{1}{1 + x^2}, \]

interpolating the function at 13 equally spaced points between -5 and 5. (In MATLAB, you can set the points with the command \( x = [-5:5/6:5] \); and you can then use routine \texttt{polyfit} to find the coefficients of a 12th degree polynomial that interpolates \( f \) at those points: \( p = \texttt{polyfit}(x,1./(1+x.^2)) \). You can then use \texttt{polyval} to evaluate the polynomial at points between -5 and 5. Evaluate it at more than just the 13 interpolation points; say, \( xx = [-5:.01:5]; yy = \texttt{polyval}(p,xx) \). Turn in a plot of \( f \) and the polynomial interpolant on the same graph.

(b) Write down the second degree (\( n = 2 \)) Bernstein polynomial,

\[ B_n(x; f) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k}, \]

for the function \( f \) in part (a). You will first have to map the interval \([-5, 5]\) to \([0, 1]\) so that you get a function whose values on this interval match those of \( f \) on \([-5, 5]\). Then evaluate \( B_2 \) for this function. Sketch the graph of the function on \([0, 1]\) and sketch the Bernstein polynomial on this same interval.

2. Let \( f \) be an \textit{even} function in \( C[-1, 1] \); i.e., \( f(x) = f(-x) \forall x \in [-1, 1] \). The Weierstrass approximation theorem tells us that \( f \) can be approximated to arbitrary accuracy in the \( \infty \)-norm by a polynomial. Show that \( f \) can be approximated to arbitrary accuracy in the \( \infty \)-norm by an \textit{even} polynomial (a polynomial of the form \( \sum_{j=0}^{n} c_j x^{2j} \)).

3. Sometimes, instead of approximating a continuous function by a single polynomial of high degree, one approximates it by \textit{piecewise polynomials} of low degree. For example, if \( f \) is continuous on the interval \([0, 1]\), one might divide the interval \([0, 1]\) into \( n \) subintervals, each of width \( h = \frac{1}{n} \), and approximate \( f \) by a function that matches \( f \) at the endpoints of each subinterval and is linear within each subinterval, as pictured below:
(a) Suppose the interval \([0, 1]\) has been divided into \(n\) equal subintervals, with endpoints \(x_0 = 0, x_1 = 1/n, \ldots, x_{n-1} = (n-1)/n, x_n = 1\), as described above. Consider the “hat” functions:

\[
\varphi_j(x) = \begin{cases} 
\frac{x-x_{j-1}}{h}, & x_{j-1} \leq x \leq x_j \\
\frac{x_{j+1}-x}{h}, & x_j \leq x \leq x_{j+1} \\
0, & \text{otherwise}
\end{cases}, \quad j = 1, \ldots, n-1.
\]

Show that any continuous piecewise linear function \(\varphi\) on this grid with \(\varphi(0) = \varphi(1) = 0\) can be written as a linear combination of the \(\varphi_j\)'s; that is, determine values \(c_1, \ldots, c_{n-1}\) such that \(\varphi(x) = \sum_{j=1}^{n-1} c_j \varphi_j(x)\).

(b) Show that the set of continuous piecewise linear functions is dense in \((C([0, 1], \| \cdot \|_\infty)); \text{i.e., every function } f \text{ in } C([0, 1]) \text{ can be approximated arbitrarily well in the } \infty\text{-norm by a continuous piecewise linear function.} \) [Hint: Since \(f\) is continuous on the compact set \([0, 1]\), it is uniformly continuous. That is, for any \(\epsilon > 0\) there exists \(\delta > 0\) such that whenever \(x, y \in [0, 1]\) and \(|x - y| < \delta\), we have \(|f(x) - f(y)| < \epsilon\). Try approximating \(f\) by its piecewise linear interpolant on a grid with spacing \(h < \delta\).]

4. Let \(A\) be an \(n\) by \(n\) matrix. Given a vector norm \(\| \cdot \|\) (a norm on \(\mathbb{R}^n\) or \(\mathbb{C}^n\), like the Euclidean norm or the 1-norm or the \(\infty\)-norm), one defines the corresponding operator norm of \(A\) to be \(\|A\| = \sup\|v\|_1 \|Av\|\). Verify that this defines a norm on the set of \(n\) by \(n\) matrices, and explain why sup could be replaced by max in the definition.

To solve a large linear system \(Ax = b\) (where the nonsingular \(n\) by \(n\) matrix \(A\) and the \(n\)-vector \(b\) are given), the following iterative procedure is sometimes used. First, write \(A\) as \(A = M - N\), where \(M\) approximates \(A\) but is easier to invert than \(A\). (For example, \(M\) might be the diagonal of \(A\) or the lower triangle of \(A\).) Starting with an initial guess \(x^{(0)}\) for the solution, form successive approximations satisfying

\[Mx^{(k+1)} = Nx^{(k)} + b, \quad k = 0, 1, \ldots.\]
Thus, $x^{(k+1)} = g(x^{(k)})$, where $g : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $g(v) = M^{-1}Nv + M^{-1}b$. Show that this iteration converges to the unique solution $x = A^{-1}b$ if $\|M^{-1}N\| < 1$. 