Sample Solutions for Applied Analysis Midterm

1. Let \( C^1([0, 1]) \) denote the set of functions \( f : [0, 1] \to \mathbb{R} \) whose first derivative \( f' \) is continuous on \([0, 1]\). Define a norm on this space by

\[
\|f\|_{C^1} = \max_{0 \leq x \leq 1} |f(x)| + \max_{0 \leq x \leq 1} |f'(x)| = \|f\|_\infty + \|f'\|_\infty.
\]

(a) Show that \( \| \cdot \|_{C^1} \) is a norm on \( C^1([0, 1]) \).

i. \( \|f\|_{C^1} \geq 0 \) since it is a sum of maxes of absolute values, which are all greater than or equal to 0. \( \|f\|_{C^1} = 0 \) if and only if \( f(x) = 0 \) and \( f'(x) = 0 \) for all \( x \in [0, 1] \), if and only if \( f \equiv 0 \).

ii. For \( \alpha \) a scalar, \( \|\alpha f\|_{C^1} = \max_{0 \leq x \leq 1} |\alpha f(x)| + \max_{0 \leq x \leq 1} |(\alpha f)'(x)| = |
\alpha| \cdot \max_{0 \leq x \leq 1} |f(x)| + |\alpha| \cdot \max_{0 \leq x \leq 1} |f'(x)| \) (since \( (\alpha f)' = \alpha f' \)) = \( |\alpha| \cdot \|f\|_{C^1} \).

iii. \( \|f + g\|_{C^1} = \max_{0 \leq x \leq 1} |f(x) + g(x)| + \max_{0 \leq x \leq 1} |f'(x) + g'(x)| \) (since \( (f + g)' = f' + g' \)) \leq \max_{0 \leq x \leq 1} (|f(x)| + |g(x)|) + \max_{0 \leq x \leq 1} (|f'(x)| + |g'(x)|) \leq \max_{0 \leq x \leq 1} |f(x)| + \max_{0 \leq x \leq 1} |g(x)| + \max_{0 \leq x \leq 1} |f'(x)| + \max_{0 \leq x \leq 1} |g'(x)| = \|f\|_{C^1} + \|g\|_{C^1} \).

Thus \( \| \cdot \|_{C^1} \) satisfies all of the requirements of a norm on \( C^1([0, 1]) \).

(b) Show that the set of polynomials is dense in \( (C^1([0, 1]), \| \cdot \|_{C^1}) \). [Hint: Try applying the Weierstrass Approximation Theorem to \( f' \).]

Given any \( \epsilon > 0 \), the Weierstrass Approximation Theorem tells us that we can approximate the continuous function \( f' \) to within \( \epsilon/2 \) by a polynomial \( g \); that is, \( \|f' - g\|_\infty \leq \epsilon/2 \). Let \( p(x) = \int_0^x g(t) \, dt + f(0) \). Then \( p \) is a polynomial and \( p' = g \). Since \( f(x) = \int_0^x f'(t) \, dt + f(0) \), we can write \( |f(x) - p(x)| = |\int_0^x (f'(t) - g(t)) \, dt| \leq \int_0^x |f'(t) - g(t)| \, dt \leq \epsilon/2 \), since \( |f'(t) - g(t)| \leq \epsilon/2 \) for all \( t \in [0, 1] \). Taking the max over \( x \in [0, 1] \), we have \( \|f - p\|_\infty \leq \epsilon/2 \), and since also \( \|f' - p'\|_\infty \leq \epsilon/2 \), it follows that \( \|f - p\|_{C^1} \leq \epsilon \). Therefore the set of polynomials is dense in \( (C^1[0, 1], \| \cdot \|_{C^1}) \).
2. A Volterra integral equation of the second kind for an unknown function \( f : [a, b] \to \mathbb{R} \)
is an equation of the form
\[
f(x) - \int_a^x k(x, y)f(y) \, dy = g(x), \quad x \in [a, b],
\]
where \( k : [a, b] \times [a, b] \to \mathbb{R} \) and \( g : [a, b] \to \mathbb{R} \) are given functions. Suppose that \( k \) and \( g \) are continuous and that
\[
\sup_{a \leq x \leq b} \int_a^x |k(x, y)| \, dy < 1.
\]
Show that this equation has a unique solution \( f \in C([a, b]) \).

Define
\[
(Tf)(x) = g(x) + \int_a^x k(x, y)f(y) \, dy.
\]

Will show that \( T \) is a contraction mapping from the complete metric space \((C[a, b], \| \cdot \|_\infty)\) into itself and therefore by the Contraction Mapping Theorem it has a unique fixed point; that is, the integral equation has a unique solution \( f \in C([a, b]) \).

To see that \( T \) maps \( C[a, b] \) into \( C[a, b] \), note that for any \( x_0 \in [a, b] \), since \( g \) is continuous, \( \lim_{x \to x_0} g(x) = g(x_0) \); since \( \int_a^x \) is continuous as long as the integrand is bounded, \( \lim_{x \to x_0} \int_a^x f(y) \, dy = \int_a^{x_0} f(y) \, dy \); and since \( k \) is continuous in \( x \) and for each \( x \), \( k(x, y)f(y) \) is continuous and hence uniformly continuous in \( y \) on \([a, b]\), we have \( \lim_{x \to x_0} \int_a^x k(x, y)f(y) \, dy = \int_a^{x_0} k(x_0, y)f(y) \, dy \). Thus \( \lim_{x \to x_0} (Tf)(x) = (Tf)(x_0) \), so \( Tf \in C[a, b] \).

Now to see that \( T \) is a contraction on \( C([a, b]) \), note that if \( f \) and \( h \) are in \( C[a, b] \), then
\[
|(Tf)(x) - (Th)(x)| = \left| \int_a^x k(x, y)(f(y) - h(y)) \, dy \right| \\
\leq \int_a^x |k(x, y)| \cdot |(f(y) - h(y))| \, dy \\
\leq \|f - h\|_\infty \int_a^x |k(x, y)| \, dy.
\]

Taking the sup over \( x \in [a, b] \),
\[
\|Tf - Th\|_\infty \leq \|f - h\|_\infty \cdot \sup_{x \in [a, b]} \int_a^x |k(x, y)| \, dy,
\]
and since the last factor is strictly less than 1, it follows that \( T \) is a contraction on \( C[a, b] \). Therefore the integral equation has a unique solution in \( C[a, b] \).
3. Let \( f(t, u) \) be continuous on \([0, 1] \times \mathbb{R}\) (mapping into \(\mathbb{R}\)) and satisfy the **generalized Lipschitz condition**

\[
|f(t, u) - f(t, v)| \leq L(t)|u - v| \quad (\forall t \in [0, 1]) \quad (\forall u, v \in \mathbb{R}),
\]

where \( L(t) \geq 0 \) and \( L \) is continuous on \((0, 1]\) but possibly unbounded near \( t = 0 \). Show that if \( \int_0^1 L(t) \, dt < \infty \), then the initial value problem \( u'(t) = f(t, u(t)), \, u(0) = u_0 \), has at most one solution on \([0, 1]\). [Hint: Suppose \( u \) and \( v \) are two solutions. Define \( w(t) = (u(t) - v(t))^2 \), and use Gronwall’s inequality on intervals starting just to the right of \( t_0 = 0 \).]

Suppose \( u \) and \( v \) are two solutions. Define \( w(t) = (u(t) - v(t))^2 \). Then \( w'(t) = 2(u(t) - v(t))(u'(t) - v'(t)) \) and since \( |u'(t) - v'(t)| = |f(t, u(t)) - f(t, v(t))| \leq L(t)|u(t) - v(t)| \), we have

\[
w'(t) \leq 2L(t)w(t).
\]

We could almost apply Gronwall’s inequality now to say that \( w(t) \leq e^{2 \int_0^t L(s) \, ds}w(0) \), but the problem is that \( L(t) \) is not continuous (in fact it can be infinite) at \( t = 0 \), so we need to move a little away from \( t = 0 \) in order for Gronwall’s inequality to be valid. Since \( u \) and \( v \) are continuous and hence \( w \) is continuous, given any \( \epsilon > 0 \), we can find \( \delta > 0 \) such that \( |w(t) - w(0)| = w(t) \leq \epsilon \) whenever \( 0 \leq t \leq \delta \). Gronwall’s inequality is valid on the interval \([\delta, 1]\), so we can write

\[
w(t) \leq e^{2 \int_\delta^t L(s) \, ds}w(\delta) \leq e^{2 \int_0^1 L(s) \, ds} \epsilon, \quad \forall t \in [\delta, 1].
\]

Since \( e^{2 \int_0^1 L(s) \, ds} \) is finite, by choosing \( \delta \) sufficiently small, we can make the bound on \( w(t) \) as small as we like, both in the interval \([0, \delta]\) and in the interval \([\delta, 1]\). Thus we conclude that \( w(t) \equiv 0 \) and there is at most one solution to the IVP.