Sample Solutions for Practice Problems for Final (Dec. 12, 2:30-4:20)

1. For an \( n \times n \) matrix \( A \), the operator norm corresponding to the \( 1 \)-norm and the \( \infty \)-norm for \( n \)-vectors is \( \max_{\|v\|_p=1} \|Av\|_p \), \( p = 1, \infty \). Recall that the \( 1 \)-norm of a matrix is the maximum absolute column sum and the \( \infty \)-norm of a matrix is the maximum absolute row sum. Determine the smallest constant \( C_{1,\infty} \) for which \( \|A\|_1 \leq C_{1,\infty} \|A\|_\infty \) for all \( A \in \mathbb{C}^{n \times n} \) and the smallest constant \( C_{\infty,1} \) for which \( \|A\|_\infty \leq C_{\infty,1} \|A\|_1 \) for all \( A \) in \( \mathbb{C}^{n \times n} \). In each case, show that the inequality holds for all \( A \) and construct a (nonzero) matrix \( A \) for which equality holds.

What (more general) theorem guarantees the existence of such constants?

\[
\|A\|_1 \leq n \|A\|_\infty, \text{ since if the } j\text{th column of } A \text{ is the one with the maximum absolute value, and if } a_{ij} \text{ is the entry in that column with largest absolute value, then } \|A\|_1 = \sum_{k=1}^{n} |a_{kj}| \leq n |a_{ij}|, \text{ while } \|A\|_\infty \geq \sum_{k=1}^{n} |a_{ik}| \geq |a_{ij}|.
\]

Therefore \( \|A\|_1 \leq n \|A\|_\infty \). We will get equality if \( A \) has only one nonzero column and each entry in that column has the same absolute value, say, 1. Then \( \|A\|_1 = n \) and \( \|A\|_\infty = 1 \).

We get the same constant in the other direction, \( \|A\|_\infty \leq n \|A\|_1 \). Since \( \|A\|_\infty = \|A^T\|_1 \) and \( \|A\|_1 = \|A^T\|_\infty \), we can just apply the above to \( A^T \).

The theorem that guarantees the existence of such constants is the one that says: In a finite dimensional vector space, all norms are equivalent. The space of \( n \) by \( n \) matrices is a vector space of dimension \( n^2 \).

2. Let \( V = P_1(\mathbb{R}) \) (i.e., the space of polynomials of degree one or less over the reals). For \( p \in V \), define elements \( f \) and \( g \) of \( V^* \) by

\[
f(p) = \int_0^1 p(t) \, dt, \quad g(p) = \int_0^2 p(t) \, dt.
\]

Prove that \( \{f, g\} \) is a basis for \( V^* \) and find a basis of \( V \) for which it is the dual basis.

Since \( \dim(V^*) = \dim(V) = 2 \), it suffices to show that \( f, g \in V^* \) and \( f \) and \( g \) are linearly independent. It is clear that \( f, g \in V^* \) since \( (\forall p, q \in V)(\forall \alpha, \beta \in \mathbb{R}) \) and for \( c = 1, 2 \),

\[
\int_0^c (\alpha p(t) + \beta q(t)) \, dt = \alpha \int_0^c p(t) \, dt + \beta \int_0^c q(t) \, dt.
\]

To see that \( f \) and \( g \) are linearly independent, note that \( f(1) = \int_0^1 1 \, dt = 1 \) and \( g(1) = \int_0^2 1 \, dt = 2 \), so if \( \alpha f + \beta g = 0 \), then \( \alpha = -2\beta \). However, \( f(t) = \int_0^1 t \, dt = \frac{1}{2} \) and \( g(t) = \int_0^2 t \, dt = 2 \), so if \( \alpha f + \beta g = 0 \), \( \alpha = -4\beta \). These two conditions imply that \( \alpha = \beta = 0 \), so \( f \) and \( g \) are linearly independent and hence form a basis for \( V^* \).
To find a basis of \( V \) to which \( \{f, g\} \) is dual, look for functions \( a + bt \) and \( c + dt \) such that

\[
\begin{align*}
f(a + bt) &= a + \frac{1}{2}b = 1 \quad f(c + dt) = c + \frac{1}{2}d = 0 \\
g(a + bt) &= 2a + 2b = 0 \quad g(c + dt) = 2c + 2d = 1.
\end{align*}
\]

It is easy to see that the solution of these equations is \( a = 2, b = -2, c = -\frac{1}{2}, d = 1 \). Hence the required basis is \( \{2 - 2t, -\frac{1}{2} + t\} \).


Let \( T_n(x) = \cos(n\theta) \), where \( \cos(\theta) = x \) and \( 0 \leq \theta \leq \pi \). We wish to show that

\[
\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1 - x^2}} \, dx = 0, \quad \text{if} \quad m \neq n.
\]

Make the change of variable \( x = \cos(\theta) \). Then \( dx = -\sin(\theta) \, d\theta \), \( \sqrt{1 - x^2} = \sin \theta \), and as \( x \) goes from \(-1\) to \(1\), \( \theta \) goes from \(0\) to \(\pi\). Hence the above integral becomes

\[
-\int_{0}^{\pi} \frac{\cos(n\theta) \cos(m\theta)}{\sin \theta} \sin(\theta) \, d\theta = -\int_{0}^{\pi} \cos(n\theta) \cos(m\theta) \, d\theta.
\]

We know from a homework exercise (Exercise 7.3, p. 183) that the functions \( \{\sqrt{1/\pi}, \sqrt{2/\pi} \cos(n\theta), n = 1, 2, \ldots \} \) are an orthonormal basis for \( L^2[0, \pi] \). Hence

\[
\|T_0\| = \left( \int_{0}^{\pi} 1^2 \, dx \right)^{1/2} = \sqrt{\pi},
\]

\[
\|T_n\| = \left( \int_{0}^{\pi} \cos^2(n\theta) \, d\theta \right)^{1/2} = \sqrt{\pi/2}, \quad n = 1, 2, \ldots.
\]

4. Let \( f(t) = |t| \) for \( |t| \leq \pi \) be a continuous \( 2\pi \)-periodic function.

(a) Compute the Fourier coefficients of \( f \).

\[
\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |t| \, dt = \frac{2}{\sqrt{2\pi}} \int_{0}^{\pi} t \, dt = \frac{\pi^2}{\sqrt{2\pi}}.
\]

For \( n \neq 0 \),

\[
\hat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |t| e^{-int} \, dt = \frac{1}{\sqrt{2\pi}} \left[ -\int_{-\pi}^{0} te^{-int} \, dt + \int_{0}^{\pi} te^{-int} \, dt \right].
\]

Using integration by parts with \( u = t, \, dv = e^{-int} \, dt \), this becomes

\[
\hat{f}(n) = \frac{1}{\sqrt{2\pi}} \left[ -t \left. \frac{e^{-int}}{n} \right|_{-\pi}^{0} + \int_{-\pi}^{0} \frac{i}{n} e^{-int} \, dt + \int_{0}^{\pi} \frac{i}{n} e^{-int} \, dt \right].
\]
\[
= \frac{1}{\sqrt{2\pi}} \left[ -\pi \frac{i e^{in\pi}}{n} - \frac{1}{n^2} e^{-int} \left|_{-\pi}^{0} \right. + \pi \frac{i e^{-in\pi}}{n} + \frac{1}{n^2} e^{-int} \left|_{0}^{\pi} \right. \right]
\]
\[
= \frac{1}{\sqrt{2\pi}} \left[ -\frac{1}{n^2} (1 - e^{in\pi}) + \frac{1}{n^2} (e^{-in\pi} - 1) \right]
\]
\[
= \begin{cases} 
0 & \text{if } n \text{ is even} \\
-4/(\sqrt{2\pi}n^2) & \text{if } n \text{ is odd} 
\end{cases}
\]

(b) Use Parseval’s identity to prove that
\[
\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.
\]

Since
\[
\int_{-\pi}^{\pi} |t|^2 \, dt = 2 \int_{0}^{\pi} t^2 \, dt = \frac{2\pi^3}{3},
\]
it follows from Parseval’s identity and part (a) that
\[
\frac{2\pi^3}{3} = \frac{\pi^3}{2} + 2 \sum_{n=1,3,\ldots} \frac{8}{\pi n^4}, \quad \text{or}, \quad \sum_{n=1,3,\ldots} \frac{1}{n^4} = \frac{\pi^4}{96}.
\]

Now,
\[
\sum_{n=2,4,\ldots} \frac{1}{n^4} = \sum_{j=1}^{\infty} \frac{1}{(2j)^4} = \frac{1}{16} \sum_{j=1}^{\infty} \frac{1}{j^4},
\]
and also
\[
\sum_{n=2,4,\ldots} \frac{1}{n^4} + \sum_{n=1,3,\ldots} \frac{1}{n^4} = \sum_{j=1}^{\infty} \frac{1}{j^4}.
\]
It follows that
\[
\frac{\pi^4}{96} = \frac{15}{16} \sum_{j=1}^{\infty} \frac{1}{j^4}, \quad \text{or}, \quad \sum_{j=1}^{\infty} \frac{1}{j^4} = \frac{\pi^4}{90}.
\]

5. If \( f(t) \) is a continuous \( 2\pi \)-periodic function, show that the differential equation \( x''(t) + x(t) = f(t) \) has a \( 2\pi \)-periodic solution if and only if
\[
\int_{0}^{2\pi} f(t) \cos(t) \, dt = 0 \quad \text{and} \quad \int_{0}^{2\pi} f(t) \sin(t) \, dt = 0.
\]

If \( x(t) \) is \( 2\pi \)-periodic then it has a Fourier series expansion,
\[
\frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(jt) + \sum_{j=1}^{\infty} b_j \sin(jt).
\]
If \( x(t) \) satisfies the differential equation, then it is in \( C^2[-1,1] \) since \( f \) is continuous, and hence its Fourier series can be differentiated twice term by term:
\[
x''(t) = -\sum_{j=1}^{\infty} a_j j^2 \cos(jt) - \sum_{j=1}^{\infty} b_j j^2 \sin(jt).
\]
Inserting the Fourier series for \( x \) and \( x'' \) into the differential equation we find

\[
f(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j (1 - j^2) \cos(jt) + \sum_{j=1}^{\infty} b_j (1 - j^2) \sin(jt),
\]

which is the unique Fourier series expansion of \( f \). Since the coefficient of \( \cos(t) \) is 0 (1 - \( j^2 \) for \( j = 1 \)) and that of \( \sin(t) \) is 0, this implies that

\[
\int_{0}^{2\pi} f(t) \cos(t) \, dt = \int_{0}^{2\pi} f(t) \sin(t) \, dt = 0. \tag{1}
\]

Conversely, if \( f \) satisfies (1), then its Fourier series has the form

\[
\frac{c_0}{2} + \sum_{j=2}^{\infty} c_j \cos(jt) + \sum_{j=2}^{\infty} d_j \sin(jt).
\]

If we set \( a_0 = c_0 \), \( a_j = c_j/(1 - j^2) \), \( j = 2, 3, \ldots \), \( b_j = d_j/(1 - j^2) \), \( j = 2, 3, \ldots \), and let \( a_1 \) and \( b_1 \) be arbitrary, then \( x(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(jt) + \sum_{j=1}^{\infty} b_j \sin(jt) \) is a 2\( \pi \)-periodic solution since it is a sum of 2\( \pi \)-periodic terms and satisfies \( x''(t) + x(t) = f(t) \), as shown above.

6. Consider an **overdetermined** system of linear equations \( Ax \approx b \), where \( A \) is an \( m \) by \( n \) matrix, \( m > n \), \( b \) is a given \( m \)-vector, and we seek an \( n \)-vector \( x \) for which \( \| b - Ax \|_2 \) is as small as possible. This is called a **linear least squares** problem.

   (a) Assume that \( A \) has rank \( k < n \). Let \( q_1, \ldots, q_k \) be an orthonormal basis for \( \text{range}(A) \). Let \( Q \) be the \( m \) by \( k \) matrix whose columns are \( q_1, \ldots, q_k \). Show that the unique closest vector to \( b \) in \( \text{range}(A) \) is \( QQ^*b \).

   From the projection theorem, the unique closest vector to \( b \) in \( \text{range}(A) \) is the vector \( y_* \) in \( \text{range}(A) \) for which \( b - y_* \) is orthogonal to \( \text{range}(A) \).

   Since the columns of \( Q \) span the range of \( A \), the vector \( QQ^*b \) is in the range of \( A \), and since the columns of \( Q \) are orthonormal, \( Q^*(b - QQ^*b) = Q^*b - (Q^*Q)Q^*b = Q^*b - IQ^*b = 0 \).

   (b) Where would the Gram-Schmidt process break down if one tried to use it to orthonormalize the columns of \( A \)?

   When you encountered a column, say, column \( j \) that was a linear combination of previous columns, you would find that the vector

   \[
   \tilde{q}_j = a_j - \sum_{i=1}^{j-1} \langle q_i, a_j \rangle q_i
   \]

   would be \( 0 \). Hence when you tried to normalize to get \( q_j = \tilde{q}_j/\|\tilde{q}_j\| \), it would break down.

   (c) Show that \( A \) can be written in the form \( A = QR \), where \( Q \) is the \( m \) by \( k \) matrix described in (a) and \( R \) is a \( k \) by \( n \) matrix with \( r_{ij} = 0 \) whenever \( i > j \). Describe all solutions \( x \) of the least squares problem.
As long as the columns of \( A \) are linearly independent, you can run the Gram-Schmidt process to find the columns of \( Q \) and the entries of \( R \):

\[
r_{ij} = \langle q_i, a_j \rangle \quad \text{for} \quad i < j, \quad r_{jj} = \| \tilde{q}_j \|.
\]

If \( \tilde{q}_j = 0 \), then set \( r_{ij}, \ i < j \) as usual, but do not add a new column to \( Q \), since now we can write

\[
(a_1, \ldots, a_{j-1}, a_j) = (q_1, \ldots, q_{j-1}) \begin{pmatrix}
  r_{11} & \ldots & r_{1,j-1} & r_{1j} \\
  \vdots & \ddots & \vdots & \vdots \\
  r_{j-1,j-1} & \ldots & r_{j-1,j} & r_{jj}
\end{pmatrix}.
\]

Now continue the Gram-Schmidt process with \( a_{j+1} \). Orthogonalize it against \( q_1, \ldots, q_{j-1} \) and store the coefficients \( \langle q_i, a_{j+1} \rangle \) in rows 1 through \( j - 1 \) of column \( j + 1 \) of \( R \). If the resulting vector \( \tilde{q}_{j+1} \) is nonzero, then normalize it and append the normalized vector \( q_{j+1} \) to \( Q \) to get an \( m \) by \( j \) matrix with orthonormal columns and set \( r_{j,j+1} = \| \tilde{q}_{j+1} \| \). If \( \tilde{q}_{j+1} = 0 \), do not append any column to \( Q \) and just move on to \( a_{j+2} \). Continuing in this way, you will get an \( n \) by \( k \) matrix \( Q \) since \( \text{rank}(A) = k \) and a \( k \) by \( n \) matrix \( R \) with \( r_{ij} = 0 \) for \( i > j \).

Since \( QQ^* b \) is in the range of \( A \), the equation \( Ax = QQ^* b \) has at least one solution, and since \( QQ^* b \) is the closest vector to \( b \) in the range of \( A \), any solution to this equation is a solution to the least squares problem. We can write this equation as \( QRx = QQ^* b \), and any \( x \) that satisfies this equation must also satisfy \( Q^* QRx = Q^* QQ^* b \), or, \( Rx = Q^* b \). Conversely, any solution of \( Rx = Q^* b \) satisfies \( QRx = QQ^* b \), so the solutions of the least squares problem are the solutions of \( Rx = Q^* b \).