Sample Solutions to Practice Problems for Midterm (Nov. 2, 2011)


1. Write down the second degree Bernstein polynomial approximation to \( f(x) = |x| \) on \([-1, 1]\).

First map the interval \([-1, 1]\) to \([0, 1]\) by setting \( y = (x + 1)/2 \), so that \( x = 2y - 1 \). Define \( g(y) = f(x) = |2y - 1| \), and find \( B_2(y; g) \):

\[
B_2(y; g) = g(0) \cdot (1-y)^2 + g(1/2) \cdot 2y(1-y) + g(1) \cdot y^2 = (1-y)^2 + y^2 = 1 - 2y + 2y^2.
\]

Mapping back into \( x \) coordinates, this is

\[
1 - 2 \frac{x + 1}{2} + 2 \left( \frac{x + 1}{2} \right)^2 = \frac{1}{2} (x^2 + 1).
\]

2. Exercise 2.4 on p. 59 in text.

The Weierstrass Approximation Theorem says that any continuous function, such as \( f(x) = |x| \) on \([-1, 1]\), can be arbitrarily well approximated by polynomials. If we map the interval \([-1, 1]\) to \([0, 1]\), as in the previous problem, then, for example the Bernstein polynomials \( B_n(y; |2y - 1|) \) converge to \(|2y - 1|\) in \( C[0, 1], \| \cdot \|_\infty \). These polynomials are not only continuously differentiable but infinitely differentiable, yet their limit is continuous but has a discontinuous derivative at \( y = 1/2 \).

3. Exercise 2.9 on p. 59 in text. [Two norms \( \| \cdot \|_a \) and \( \| \cdot \|_b \) are said to be equivalent if there are positive constants \( m \) and \( M \) such that \( m\|x\|_a \leq \|x\|_b \leq M\|x\|_a \) for all \( x \).]

For \( f \in C[0, 1] \) and \( w : [0, 1] \to \mathbb{R} \) a nonnegative continuous function, define

\[
\|f\|_w = \sup_{0 \leq x \leq 1} \{w(x)|f(x)|\}.
\]

**Claim:** If \( w(x) > 0 \) for \( 0 < x < 1 \), then \( \| \cdot \|_w \) is a norm on \( C[0, 1] \).

(a) \( \|f\|_w \geq 0 \) since \( w(x) \geq 0 \) \( \forall x \in [0, 1] \) and \( |f(x)| \geq 0 \). \( \|f\|_w = 0 \) if and only if \( w(x)|f(x)| = 0 \) for all \( x \in [0, 1] \). Since \( w(x) > 0 \) for all \( x \in (0, 1) \), this could happen only if \( f(x) = 0 \) for all \( x \in (0, 1) \), and since \( f \) is continuous this implies that \( f(0) = f(1) = 0 \) as well. Thus \( \|f\|_w = 0 \) if and only if \( f \equiv 0 \).

(b) For \( \alpha \) a scalar, \( \|\alpha f\|_w = \sup_{0 \leq x \leq 1} w(x)|\alpha f(x)| = \sup_{0 \leq x \leq 1} w(x)|\alpha| |f(x)| = |\alpha| \sup_{0 \leq x \leq 1} w(x)|f(x)| = |\alpha| \cdot \|f\|_w \).
(c) For $f, g \in C[0, 1]$, \( \|f + g\|_\infty = \sup_{0 \leq x \leq 1} w(x) |f(x) + g(x)| \leq \sup_{0 \leq x \leq 1} w(x) (|f(x)| + |g(x)|) \leq \sup_{0 \leq x \leq 1} w(x) |f(x)| + \sup_{0 \leq x \leq 1} w(x) |g(x)| = \|f\|_w + \|g\|_w. \)

Thus \( \|\cdot\|_w \) satisfies all of the requirements of a norm.

If \( w(x) > 0 \) for \( 0 \leq x \leq 1 \), then \( w \) attains its infimum on \([0, 1]\) so this infimum is greater than 0; call it \( m \). Likewise, \( w \) attains its supremum on \([0, 1]\) so this supremum is finite; call it \( M \). Then we have \( \|f\|_w = \sup_{0 \leq x \leq 1} w(x)|f(x)| \leq (\sup_{0 \leq x \leq 1} w(x) \cdot (\sup_{0 \leq x \leq 1} |f(x)|) = M\|f\|_\infty. \) Also, \( \|f\|_w \geq (\inf_{0 \leq x \leq 1} w(x) \cdot (\sup_{0 \leq x \leq 1} |f(x)|) = m\|f\|_\infty. \) Thus \( \|\cdot\|_w \) is equivalent to \( \|\cdot\|_\infty \).

The norm \( \|\cdot\|_x \) corresponding to \( w(x) = x \) is not equivalent to \( \|\cdot\|_\infty \) because there is no constant \( m > 0 \) such that \( \|f\|_x \geq m\|f\|_\infty \) for all \( f \in C[0, 1] \). Whatever \( m > 0 \) we try, we can always take \( f = 0 \) on \([m/2, 1]\) and nonzero on \([0, m/2]\). Then \( \|f\|_\infty = \sup_{0 \leq x \leq m/2} |f(x)| \), while \( \|f\|_x \leq (m/2) \cdot \sup_{0 \leq x \leq m/2} |f(x)| \).

**Claim:** The metric space \((C[0, 1], \|\cdot\|_x)\) is not complete.

Let 

\[
\phi_n(x) = \begin{cases} 
    x^{-1/2} & 1/n \leq x \leq 1 \\
    n^{1/2} & 0 \leq x \leq 1/n
\end{cases}
\]

Then if \( m > n \),

\[
xf_m(x) - x\phi_n(x) = \begin{cases} 
    0 & 1/n \leq x \leq 1 \\
    x^{1/2} - xn^{1/2} & 1/m \leq x \leq 1/n \\
    x(m^{1/2} - n^{1/2}) & 0 \leq x \leq 1/m
\end{cases}
\]

Now,

\[
\sup_{0 \leq x \leq 1/m} |x(m^{1/2} - n^{1/2})| = |m^{-1/2} - n^{1/2}/m| \leq m^{-1/2} \to 0 \text{ as } m \to \infty.
\]

The maximum of \( |x^{1/2} - xn^{1/2}| \) over \( x \in [1/m, 1/n] \) occurs either at an endpoint (where both values go to 0 as \( n, m \to \infty \)) or at a point where \( d/dx((x^{1/2} - xn^{1/2})^2) = 2(x^{1/2} - xn^{1/2})(1/2)x^{-1/2} - n^{-1/2} = 0 \); i.e., at \( x = 1/(4n) \). At \( x = 1/(4n) \), we have \( |x^{1/2} - xn^{1/2}| = 1/(4\sqrt{n}) \), which also goes to 0 as \( n \to \infty \). Thus the sequence \((\phi_n)\) is a Cauchy sequence in the \( x \)-norm. But it does not converge to any continuous function on \([0, 1]\). It converges in \( x \)-norm to \( x^{-1/2} \), which is in \( C(0, 1) \) but not in \( C[0, 1] \) (and not in the equivalence class of any function in \( C[0, 1] \)).

4. Let \( \phi(x) = (x^2 + 4)/5 \). Note that \( \phi(x) = x \) if \( x = 1 \) or \( x = 4 \). Use the contraction mapping theorem to show that if one starts with any \( x_0 \in [-2, 2] \) then the iteration \( x_{k+1} = \phi(x_k) \) converges to the unique fixed point of \( \phi \) in \([-2, 2] \) (i.e., to 1).

\[ \phi \] maps the interval \([-2, 2]\) to \([4/5, 8/5] \subset [-2, 2]\). Since \([-2, 2]\) is a closed subspace of the complete metric space \( \mathbb{R} \), it is a complete metric space. Thus \( \phi \) maps a complete metric space into itself, and on \([-2, 2]\), \( \phi \) is a contraction since \( \phi'(x) = 2x/5 \) which is bounded in absolute value by \( 4/5 \). Therefore
the contraction mapping theorem tells us that \( \varphi \) has a unique fixed point in \([-2, 2]\) and starting with any \( x_0 \in [-2, 2] \), the iteration \( x_{k+1} = \varphi(x_k) \) will converge to this fixed point.

5. Exercise 3.4 on p. 79 in text.

Using the argument at the top of p. 63, with \( m = 0 \), we see that for any \( n \)
\[
d(x_n, x_0) \leq (1/(1 - c))d(x_1, x_0),
\]
so it follows (since \( \lim_{n \to \infty} x_n = x \) and \( d(\cdot) \) is a continuous function) that
\[
\lim_{n \to \infty} d(x_n, x_0) = d(\lim_{n \to \infty} x_n, x_0) = d(x, x_0) \leq \frac{1}{1 - c}d(x_1, x_0).
\]

6. Exercise 3.6 on p. 79 in text.

Define \( T : C[-a, a] \to C[-a, a] \) by
\[
Tf(x) = 1 + \frac{1}{\pi} \int_{-a}^{a} \frac{1}{1 + (x - y)^2} f(y) dy,
\]
for \( -a \leq x \leq a \).

According to Theorem 3.3, there is a unique continuous function \( f : [-a, a] \to \mathbb{R} \) satisfying \( Tf = f \) if
\[
\sup_{-a \leq x \leq a} \frac{1}{\pi} \int_{-a}^{a} \frac{1}{1 + (x - y)^2} dy < 1.
\]

Since this integral is \( \arctan(a-x) - \arctan(-a-x) \) and \( \arctan \) takes on values in \((-\pi/2, \pi/2)\), the expression on the left must be less than \((2/\pi)\arctan(2a)\), which is strictly less than 1. Thus, the integral equation has a unique continuous solution, which is bounded since it is defined on the compact interval \([-a, a]\).

This solution can be computed by starting with any initial guess, say, \( f_0(x) \equiv 0 \) and iterating according to \( f_{k+1} = T(f_k), \ k = 0, 1, \ldots \). Since \( 1/(1 + (x - y)^2) \) is nonnegative for all \( x \) and \( y \), if \( f_k \) is nonnegative then \( f_{k+1} \) will be nonnegative. Thus all of the functions \( f_k, \ k = 0, 1, \ldots \) will be nonnegative so their limit will be nonnegative.

For \( a = \infty \), the mapping \( T \) is no longer a contraction, and the contraction mapping theorem does not guarantee the existence of a unique solution. In fact, if one starts with \( f_0(x) \equiv 0 \) and iterates by setting \( f_{k+1} = T(f_k) \), then one obtains \( f_1(x) = 1, f_2(x) = 2, \ldots, f_k(x) = k, \ldots \), and this sequence does not converge. The equation might still have a unique solution, but we must use some other method to determine that.

7. Determine an interval about \( t_0 = 0 \) (if any) on which the initial value problem \( u' = \sqrt{|u|}, \ u(0) = 1 \), has a unique solution. Justify your answer by citing a theorem and find the largest interval on which the theorem guarantees a unique solution, or demonstrate that the problem has either no solution or multiple solutions on any interval about \( t_0 = 0 \).
\[ f(t, u) = \sqrt{|u|} \] is Lipschitz on an interval about \( u_0 = 1 \) of width \( 1 - \epsilon \) for any fixed \( \epsilon > 0 \). The maximum value of \( \sqrt{|u|} \) on such an interval is \( \sqrt{2 - \epsilon} \). Hence from Theorem 3.10, the solution exists and is unique for \( t \) in an interval of width \( (1 - \epsilon)/\sqrt{2 - \epsilon} \approx 1/\sqrt{2} \) about \( t_0 = 0 \).

8. Suppose \( f(t, u) \) is continuous in \( t \) and \( u \) and uniformly Lipschitz in \( u \). Suppose \( v \) and \( w \) are \( C^1 \) for \( t \geq t_0 \) and satisfy

\[
\begin{align*}
v'(t) &= f(t, v(t)) & w'(t) &\leq f(t, w(t)) \\
v(t_0) &= v_0 & w(t_0) &\leq v_0
\end{align*}
\]

Show that \( w(t) \leq v(t) \) for all \( t \geq t_0 \).

Proof by contradiction: Suppose \( w(T) > v(T) \) for some \( T > t_0 \). Since \( v \) and \( w \) are continuous, there is a point \( t_1 \) between \( t_0 \) and \( T \) where \( v(t_1) = w(t_1) \) and \( w(t) > v(t) \) for \( t \in (t_1, T] \). For \( t \in [t_1, T] \), we have \( w(t) - v(t) = |w(t) - v(t)| \) and so, if \( f \) has Lipschitz constant \( L \),

\[
(w - v)'(t) \leq f(t, w(t)) - f(t, v(t)) \leq L|w(t) - v(t)| = L(w - v)(t).
\]

By Gronwall’s inequality (applied to \( w - v \) on \([t_1, T], \) with \( (w - v)(t_1) = 0 \), \( (w - v)(t) \leq 0 \) on \([t_1, T], \) which is a contradiction since we assumed \( w(T) > v(T) \). Therefore there can be no such point \( T \) where \( w(T) > v(T) \); that is, \( w(t) \leq v(t) \) for all \( t \geq t_0 \).