Lecture Notes 2 – Induction
CSS 549 – Algorithm Design and Analysis – Professor Clark F. Olson

Introduction to induction
Consider the hypothesis: $2 + 4 + 6 + 8 + \ldots + 2n = n (n + 1)$ for all $n > 0$. Let’s look at the first few cases:

- **n = 1**: Left side: 2
  Right side: $1 (1 + 1) = 2$

- **n = 2**: Left side: $2 + 4 = 6$
  Right side: $2 (2 + 1) = 6$

- **n = 3**: Left side: $2 + 4 + 6 = 12$
  Right side: $3 (3 + 1) = 12$

- **n = 4**: Left side: $2 + 4 + 6 + 8 = 20$
  Right side: $4 (4 + 1) = 20$

The hypothesis is true for the first four cases. Is it true for all cases with $n > 0$? You can’t prove that a statement like this is true by example, but you can prove it false if you find a counter-example. One method to prove statements like this is to use mathematical induction.

The principle of mathematical induction
The principle of mathematical induction is an axiom of mathematics used to prove hypotheses that state something is true for all integers greater than zero (or some other number). A simple form can be stated as follows:

Suppose we have a statement $S(n)$ that is either true or false for all integers, $n > 0$.

The statement is true for all $n > 0$, if:

1. The statement $S(1)$ is true.
2. Assuming $S(n)$ is true for every $0 < n \leq k$, then $S(k + 1)$ is true.

The first requirement is called the basis step or base case. The second requirement is the inductive step, composed of the inductive hypothesis and the inductive conclusion. Let’s use this to prove our example above. The base case ($n = 1$) is true as demonstrated above in the introduction. Now, we **assume that** $2 + 4 + 6 + 8 + \ldots + 2n = n (n + 1)$ is true for $0 < n \leq k$. We need to prove that the statement is true for $n = k + 1$, that is:

$$2 + 4 + 6 + 8 + \ldots + 2k + 2(k + 1) = (k + 1) (k + 1 + 1) = (k + 1)(k + 2) = k^2 + 3k + 2$$

Since we are assuming that the statement is true for $n = k$, we have:

$$2 + 4 + 6 + 8 + \ldots + 2k = k (k + 1)$$

Substituting this into the inductive conclusion, we get:

$$k(k + 1) + 2(k + 1) = k^2 + 3k + 2$$
$$k^2 + 3k + 2 = k^2 + 3k + 2$$

This proves the inductive step and, thus, the statement must be true for all $n > 0$. 
Intuition

Induction looks like black magic. Are we assuming that something is true to prove that it is true? No. Think of it this way. We want to prove $S(n)$ is true for $n > 0$.

First, we prove $S(1)$ is true.
If we prove $S(1)$ implies $S(2)$, then $S(2)$ is true.
If we prove $S(2)$ implies $S(3)$, then $S(3)$ is true.
And so on. The inductive step proves all of statements of this type simultaneously:
If we prove $S(0)$ is true and $S(k)$ implies $S(k+1)$ for every $k > 1$, then every $S(k+1)$ is true (no matter what $k$ is)!
Thus, all $S(n)$ are true for $n > 0$.

Inequality example

Induction can be applied to problems where the base case is not $n = 0$. Let’s prove that $2^n > n^2$ for all $n >= 5$. Base case ($n = 5$):
Is $2^5 > 5^2$?
Yes, $32 > 25$.

Inductive case: Assume that $2^n > n^2$ for all $5 <= n <= k$, prove $2^{k+1} > (k + 1)^2$.

$2^{k+1} = 2 * 2^k > 2 * k^2 = k^2 + k^2 > k^2 + 2k + 1 = (k + 1)^2$.

Note that I didn’t prove that $k^2 > 2k + 1$ for $k >= 5$, which is necessary in the above proof. This can be easily demonstrated by another induction proof:

Base case ($n = 5$):
Is $5*5 > 2*5+1$?
Yes, $25 > 11$.

Inductive case: Assume that $k^2 > 2k + 1$ for all $5 <= n <= k$, prove $(k + 1)^2 > 2(k + 1) + 1$.

$(k + 1)^2 = k^2 + 2k + 1 > 2k + 1 + 2k +1 = 4k +2 > 2k +2$.

Divisibility problem

Is $n^3 – n$ divisible by 6 for $n > 0$? In other words, is $n^3 – n$ mod 6 = 0 for $n > 0$?

Basis step: For $n = 1$, $n^3 – n = 0$, which is divisible by 6.

Inductive step: Assume $n^3 – n$ is divisible by 6 for $0 < n <= k$. In particular, $k^3 – k$ is divisible by 6.

Prove: $(k+1)^3 – (k+1)$ is divisible by 6.

$(k+1)^3 – (k+1) = k^3 + 3k^2 +3k – k$. Now we can subtract a number that is divisible by 6 without changing whether this number is divisible by six. Therefore, we can subtract $k^3 – k$ using our inductive hypothesis, yielding:

$3k^2 + 3k = 3(k+1)$. When is a number divisible by 6? It has to be both divisible by 2 and divisible by 3. $3k(k+1)$ is divisible by 3 (because of the leading 3). Is it divisible by 2? Yes. Either $k$ or $k+1$ must be an even number and thus divisible by 2. When we multiply a number divisible by 3 times one divisible by 2, we always get a number divisible by 6. Thus, $n^3 – n$ is divisible by 6 for all $n > 0$. 
Postage problem (optional)
For some problems, we can’t just use the next smaller problem S(k) to prove S(k+1) (we need to use S(k-1) or other values of n). For example, let’s prove that we can make postage of n cents using just 2 cent stamps and 5 cent stamps, where n > 3.

Basis step: We can make 4 cents postage using two 2 cent stamps.
Inductive step: Assume we can make postage of n cents where 3 < n <= k. Prove we can make k+1 cents postage. Well, if we can make k-1 cents postage, then we just need to add a 2 cent stamp. Are we done? Not quite. If k = 4, then we are assuming that we can make 3 cents postage! Our inductive hypothesis says we can only assume that we can make n > 3 cents postage. To correct this situation we also need to show that we can make 5 cents postage, so that we don’t need to depend on making 3 cents postage. Of course, we can make 5 cents postage, we use a single 5 cent stamp…

Tiling problem
Now let’s examine a completely different type of problem. This is a problem with triominoes. A triomino is a block like a domino, except it has three squares at a right angle (not all in a row). The problem is: Given a board of size 2^n x 2^n (n > 0), with exactly one block missing (it could be any block), can the board be tiled with triominoes so that every block (except the missing one) is covered by exactly one triomino and each triomino is completely within the board. Interestingly, the answer is yes and we will prove it.

The base case is n = 1. In this case, the board with one block missing always fits just one triomino and this tiles all of the remaining blocks.

For the inductive case, we will assume that we can tile a 2^k x 2^k board with one block missing. We need to show that we can tile a 2^{k+1} x 2^{k+1} board with one block missing. Solution: Place a triomino in the center of the board such that it has one block in each of the three quadrants that don’t contain the missing block. The board can now be divided into four smaller boards, each of which is a 2^k x 2^k board with one block missing. We assumed that we knew how to tile these, so we must be able to tile the entire board. Amazingly, this is all we need to prove that any such board can be tiled!

At this point, let’s draw an 8x8 board and apply the solution using recursion, which is closely related to induction:

Practice problems (optional):
Prove 3^{2n} – 1 is divisible by 8 for n > 0. (This one is tricky.)

Prove: \[ \sum_{i=1}^{n} 2i - 1 = n^2 \] for n > 0.