Investigation #2 – A Survey of Kid Kalculus: The Walking Game

Introduction.
In this Investigation we will study the main concepts of calculus as they are realized in the discrete phenomenon of a person walking along a straight line and recording her position at equal intervals of time. Before getting into the details of the various concepts we will study, we first consider a simple visual approach to recording this person’s motion.

Imagine that we have a 20-foot straight line in a large room and that the person doing the walking, called Wanda, has fifteen small beanbags in her hands. As she walks, someone else calls out the word “drop” at 2-second intervals. At each call of the word “drop,” Wanda drops one of the small beanbags. After 30 seconds, the beanbags lie on the floor along the straight line and we have a representation of Wanda’s motion like the Beanbag diagram shown below.

1. To get a sense of how this representation works, give a play–by–play description of Wanda’s motion. Did she start out fast and then go slow, or something else? What happened after the first few drops? Was there ever a time when Wanda slowed down and then speeded up? Was there a time when she went at a constant speed?

2. Now imagine a second walker, called William, moving along the same straight line as Wanda, but slightly to her right. William’s beanbag drops are indicated in the Beanbag diagram shown above as diamonds. First, give the same kind of play–by–play description of William’s motion as you gave for Wanda’s. Then, give a play–by–play description of the motion of the two walkers in relations to one another.

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1 This approach to representing and reasoning about motion is found in very creative material intended for students in the fifth grade: Tierney, C., Nemirovsky, R., & Noble, T. (1996). Patterns of change: Tables and graphs (Curricular unit for grade 5 in Investigations in Number, Data, and Space). Palo Alto, California: Dale Seymour Publications

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another, as if they were in a race. Who was ahead at various times; were they getting further apart or closer together? Who won, etc?

As you can see, these Beanbag diagrams actually involve two kinds of numerical information—each walker’s Positions, as measured from the start, and the distances between his or her successive Positions, what we will call their Step-sizes. You almost certainly used both kinds of information as you gave your play-by-play descriptions, although you may not have been aware of this. The walker’s Positions tell where he or she is at any given time. They tell us who is ahead at any particular time, who won, etc. The Step-size is what tells us how fast a walker is going at any given time. For instance, we can state the general rule that “smaller Step-size indicates slower motion” and “larger Step-size indicates faster motion.” A sequence of Step-sizes that are of equal magnitude indicates motion at a constant speed.2

But now suppose you wanted to make a numerical record of the motion of the walkers. Even though both kinds of information, Position and Step-size, are in a Beanbag diagram, you would not need to record both, since you can always compute one kind from the other. If you had a complete record of Positions, you could subtract successive Positions to get Step-sizes. If you had a complete record of Step-sizes and you wanted to know how far the walker went when taking a certain number of steps, you would add the sizes of the steps. Thus, the walker’s position after a given number of steps, would be computed by adding all the Step-sizes, from the start, up to that number of steps. This kind of addition, adding things from the start, is called accumulation.

The fact that you can always compute one of these types of information from the other is central to calculus. Some might say that it is the biggest idea in calculus. In the case of continuous motion, it says that you can compute speed from distance and distance from speed. The main difference between the continuous case and the Walking Game is that the notions of “subtraction” and “accumulation” are much more complicated and subtle to define in the continuous case. In calculus, they are called “differentiation” and “integration,” but these are still related to subtraction and accumulation. And they are inverses of one another in very much the same way subtraction and accumulation are—which we will need to spell out more clearly.

This duality of these two kinds of information (Position and Step-size in the discrete case; Position and Speed in the continuous case) and the operations for going between them (subtraction and accumulation in the discrete case; differentiation and integration in the continuous case) was systematically employed by Newton and Leibniz when they developed calculus in the 17th century. It is interesting to note that, in inventing the

2 Of course, all we know about these walkers’ motion is their Positions at the end of each 2-second interval. It could be that in the course of such an interval, he or she speeded up or slowed down enormously. But, as is usually the case with such data, we assume that the patterns of motion are relatively stable between the beginning and the end of each 2-second intervals.
ideas of calculus, Newton was thinking primarily about motion, while Leibniz began thinking about what we now call calculus by exploring relations among number sequences. Almost everything we do in this material relates in one way or another to this duality. We begin to study it in the case of the Walking Game, using graphs, tables, and formulas, and we extend our understanding of it, as we move on through the other Investigations. The statement of this duality in the continuous case is called The Fundamental Theorem of Calculus.

In this Investigation, we will develop the ideas of calculus in this context of the Walking Game by considering five Basic Tasks that emerge in the analysis of this kind of phenomenon of change. These Basic Tasks are central to the development of calculus in any situation, whether discrete or continuous, whatever the representation. It is important that you understand the overall structure behind these Tasks and how the ideas of calculus emerges from them. In this Investigation, you will take on all these Tasks, as they are realized in the context of the Walking Game. In the next Investigation, #3, you will explore in much greater depth the next two Basic Tasks. In each of the last three Investigations, you take up one of the Tasks.

**Overview of the Five Basic Tasks.**

One way to view the overall situation of the Walking Game is that we will be given data about a particular trip in terms of one kind of information, Position or Step-size, and we want to be able to compute the data about the same trip in terms of the other kind of information, Step-size or Position. But this data, whether it is about Position or Step-size, may be represented in one of three possible ways: sequence (or table), graph, or formula. When we make our computations, we want the new data to be in the same form of representation as the data we were given. That is, we want to get a sequence directly from a sequence, a graph directly from a graph, and a formula directly from a formula. So, in effect, our goal is to have six kinds of computation: two between Position and Step-size when we are working in sequences, two in graphs, and two in formulas. We find, however, that some of these computations are the same whether we are working with sequences or graphs. We also find that because of the mathematical complexities involved, there is a certain logical order we must follow. Thus we arrive at the five Basic tasks described below.

I will first give a brief description of the five Basic Tasks and the logic that connects them. After that, you will answer specific questions to get a clearer sense of what is
involved in each Basic Task. You will then pull together and reflect on this structure in your Project Report.

The Five Basic Tasks
Our first and simplest task is to learn how to compute the sequence of Step-sizes when we are given a sequence of Positions. You probably did this automatically in interpreting the Beanbag diagrams above.

Task #1. The Derived Operator: Compute Step-size sequence/graph from the Position sequence/graph.

We begin with a trip defined by a Position sequence. How to describe the operation on this sequence that yields the Step-size sequence? Similarly, we are given a graph of Positions. How to describe the operation on this graph that gives the graph of Step-sizes?

We then want to learn more about this operator in which we get Step-size information from Position information. This can best be done via graphs.

Task #2. Using Shape to Guess Backwards in the Derived Operator. Guessing Backwards from the Step-size graph to Position graph.

We are given a graph of Step–sizes. We want to build up a set of procedures for sketching a fairly accurate version of the graph of Positions – using the visual features of the graph. How do features of the shape of the graph of Step-sizes tell us about features of the graph of Positions? Which features of the graph of Step-sizes are significant? Which features are prominent but not significant?

Now we address the situation in which we are given a formula for a Position sequence, and we want to quickly and easily obtain a formula for the corresponding step-size sequence. This is closely related to the process of taking a derivative.

Task #3. The Derived Operator and Formulas. Given a formula for a Position sequence/graph, determine the formula for the Step-size sequence/graph.

Suppose we have a formula for the Position sequence or graph. We want to quickly and easily write the formula for the Step-size sequence or graph. How to do it? How to generate rules for doing it?

In Basic Task #2, you learn how to make guesses or inferences about the shape of a Position graph from the shape of a Step-size graph. But it should be possible to directly compute the Position sequence or graph from the Step-size sequence or graph. This is a fairly straightforward task in the situation of the Walking Game. We want to learn to do it well enough to be able to handle much more complicated situations.

Task #4. The Reverse Operator – Accumulation. From the Step-size sequence/graph to the Position sequence/graph.
We are given a Step-size sequence and want to define a rule (or Operator) for producing the corresponding Position sequence. We want to be able to do the same for graphs.

Finally, we get back to formulas and the task of starting with a formula for a Step-size sequence or graph and obtaining the formula for the Position sequence or graph.

**Task #5. The Reverse Operator Using Formulas.** Given a formula for a Step-size sequence/graph, determine the formula for the corresponding Position sequence/graph.

We are given a formula for a Step-size sequence or graph. How to generate a formula for the corresponding Position sequence or graph?

Your work

Now you will go through these five Basic Tasks, answering very specific questions, in order to become much more familiar with the Basic Tasks and the overall structure connecting them. Some of these questions are relatively easy to answer, especially at first. However, as you work through them, you will probably find that they become more difficult. You should make sure you share your work with the others in your group, in order to be sure that you understand the general ideas here and how they fit together.

**Task #1. The Derived Operator:** Compute Step-size sequence/graph from the Position sequence/graph.

We begin with a trip defined by a Position sequence. How to describe the operation on this sequence that yields the Step-size sequence? Similarly, we are given a graph of Positions. How to describe the operation on this graph that gives the graph of Step-sizes?

a. Here are the Position sequences for two different trips. For each trip, describe the overall pattern of motion and then write out the Step-size sequence.

   0, 5, 10, 15, 16, 18, 21, 24, 24, 24, 24, 30, 35, 39, 42, 44
   0, 10, 19, 27, 34, 32, 30, 28, 28, 28, 31, 35, 40, 46, 45, 44

b. Now suppose you are given a trip that has as its Position sequence the values  
   \(a_1, a_2, a_3, a_4, a_5, a_6, \ldots, a_n\)  
   Using the letters, \(a_1, a_2, a_3, \ldots\), write out the terms of the Step-size sequence for this trip.

c. The two graphs on the next page are of Positions for two Walking Game trips. In each case, describe the motion of the walker; i.e., give a “play-by-play description” of what you would see the walker do. Then, for each trip make a graph of Step-sizes for the trip.
d. Now imagine that you had a graph of Positions for a “general trip,” in the same sense that you had, in b. above, a Position sequence for a “general trip.” Thus, it might have shape, but no particular numbers. Give a procedure you would use on this graph in order to produce the graph of Step-sizes.

![Graph of Step-sizes](image1)

![Graph of Positions](image2)

**Task #2. Using Shape to Guess Backwards in the Derived Operator.** Guessing Backwards from the Step-size graph to Position graph.

We are given a graph of Step-sizes. We want to build up a set of procedures for sketching a fairly accurate version of the graph of Positions – using the visual
features of the graph. How do features of the shape of the graph of Step-sizes tell us about features of the graph of Positions? Which features of the graph of Step-sizes are significant? Which features are prominent but not significant?

a. First start by looking at the two graphs you worked with in Task #1, and begin to form some principles or rules for how the shape of a Position graph relates to characteristics of motion. In order to refine your rules, you might also consider the two graphs below.

b. Below are some sketches of graphs of Step-sizes for various trips. [There are 15 separate steps in each. I have not drawn the markers, in order to keep things simple.] Again, give a play-by-play description of the walker’s motion. Tell how various features of the graph show up in the walker’s motion.
c. On the basis of what you did in b., begin to generate some rules for how features of the shape of the Step-size graph correspond (or don’t correspond) to features of shape of the Position graph. Are there features of the shape of the Step-size graph that you would expect to show up as aspects of motion or in features in the Position graph that don’t do so? For instance, would it make much difference in the overall shape of the Position graph if the straight lines I drew in the graphs just above were bent, even quite a bit? For instance, what if the first were shaped like the graph to the right?

Task #3. The Derived Operator and Formulas. Given a formula for a Position sequence/graph, determine the formula for the Step-size sequence/graph.

Suppose we have a formula for the Position sequence or graph. We want to quickly and easily write the formula for the Step-size sequence or graph. How to do it? How to generate rules for doing it?

a. A walker moves along the line in such a way that she starts at 0, is at 1 ft after the first step, is at 4 ft after step 2, is at 9 ft after step 3, ….. and is at \( n^2 \) ft after the \( n^{\text{th}} \)-step, where \( n \) goes from 0 to 15. First write out the Step-sizes for her steps #1, 2, 3, ….. Then, using the letter \( n \) write out the Step-size for the \( n^{\text{th}} \)-step. [Hint: You will find it necessary to use the formula \((a + b)^2 = a^2 + 2ab + b^2\).] Draw the Position and Step-size graphs for this walker.

b. Repeat all of what you just did in a. with a walker whose Position after the \( n^{\text{th}} \)-Step-size is given by the formula \( 20n - (1/2)n^2 \).

c. Question: Can you generalize what you’ve done here and connect it to what one does in a calculus course with formulas? For instance, what about a trip in which the \( n^{\text{th}} \)-Step-size is given by the formula \( n^3 + 4n^2 + 2 \)? On the basis of those connections, would you generalize (or speculate) what will happen with trips in the Walking Game described by any formula for Position?

Task #4. The Reverse Operator – Accumulation. From the Step-size sequence/graph to the Position sequence/graph.

We are given a Step-size sequence and want to define a rule (or Operator) for producing the corresponding Position sequence. We want to be able to do the same for graphs.
a. Here are the Step-size sequences for two different trips. For each trip, describe the overall pattern of motion and then write out the Position sequence, assuming that the walker was at 0 ft at the beginning of the trip.

\[1, 1, 1, 1, 2, 3, 4, 5, -1, -2, -3, -4, -4 \quad 1, 2, 3, 4, 5, 6, -7, -7, -6, -5, -4, -3, -2, -1\]

b. Now suppose you are given a Trip that has as its Step-size sequence the values

\[a_1, a_2, a_3, a_4, a_5, a_6, \ldots, a_n\]

Using the letters, \(a_1, a_2, a_3, \ldots\), write out the terms of the Position sequence for this trip. If you are at all familiar with summation notation, then you should start using it here. It is very helpful in problems of this kind. In general, using summation notation, we write

\[a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \ldots + a_k = \sum_{i=1}^{k} a_i\]

Again, assume that the walker was at 0 ft at the beginning of the trip.

c. Draw the Step-size graphs of the two trips whose Step-size sequence you were given in a, above. For each trip make a graph of Position for the trip, assuming that the walker was at 0 ft at the beginning of the trip. Before going on to the next question, can you see how you might define a general construction of the Position graph from the Step-size graph using graphical or visual ideas alone?

d. Now imagine that you had a graph of Step-size for a “general trip,” in the same sense that you had, in b. above (a Step-size sequence for a “general trip”). Give a procedure you would use on this graph in order to produce the graph of Position. As usual, in this Investigation, assume that the walker was at 0 ft at the start of the trip. Again, try to come up with a graphical description, not in terms of numbers or arithmetic operations.

**Task #5. The Reverse Operator Using Formulas.** Given a formula for a Step-size sequence/graph, determine the formula for the Position size sequence/graph.

We are given a formula for a Step-size sequence or graph. How to generate a formula for the corresponding Position sequence or graph?

a. We are considering the trip in which the Step-size sequence is given by the formula.

\[3, 5, 7, \ldots, 2n + 1, \text{ for } n = 1, 2, 3, \ldots 15\]  
That is, the size of the \(n^{\text{th}}\) step is \(2n + 1\) feet. First, generate the first 5 or 6 terms of this sequence and then use the kinds of approach you developed in Task #4 to generate the Position sequence for this trip. (Make the usual assumption that the walker starts at 0 ft.) Draw the graphs of Step-size and Position for this trip. Would you make a guess of what the formula for the Position sequence is? If you think you know what it is, how would you justify your answer?
b. Repeat what you did in a. for a different trip, one that has as its Step-sizes,

\[ \frac{39}{2}, \frac{37}{2}, \frac{35}{2}, \frac{33}{2}, \ldots, \frac{-n+41}{2}, \ldots, \frac{1}{2} \]

c. Question: Can you generalize what you’ve done here and connect it to what one does in a calculus course with formulas? On the basis of those connections, would you generalize (or speculate) what will happen with trips in the Walking Game described by a formula for Step-sizes?

Your Report for this Investigation.

One of the aspects of calculus that makes it so powerful is that it is so general; it applies to so many situations. It is not just about motion, but can be used to analyze phenomena in economics, statistics, and an enormous variety of physical contexts other than motion. Calculus can be applied so easily to other situations, because the ideas of the subject are developed outside any particular context; they are developed in the so-called “naked context,” of pure \( x \)'s and \( y \)'s. But so far, we have developed the ideas of the calculus of sequences only in the context of motion, of people walking along a straight line. Your overall job in writing a report about this Investigation is to work through the development of the ideas of the calculus of sequences while referring only to sequences of numbers,

\[ \{a_i\}_{i=1}^n = a_1, a_2, a_3, a_4, a_5, a_6, \ldots, a_{n-1}, a_n \]

You are not allowed to make any reference to Position, Step-size, or speed.

You will probably be surprised at how easy it is to write out some aspects of this development, while others seem rather difficult. Thinking in terms of motion not only motivates us, but helps us know what we are talking about at any given moment. For instance, in the Walking Game we might have been talking about a Position sequence and how to obtain the corresponding Step-size sequence. But now, in this development, you will simply be talking about one sequence and how to obtain another sequence from it by subtraction. Given the sequence,

\[ 1, 3, 5, 6, 9, 5, 2, 4, \]

we produce another sequence \( 2, 2, 1, 3, -4, -3, 2 \). But we also produce the sequence by accumulation, \( 1, 4, 9, 15, 24, 29, 31, 35 \).

The first thing we notice is that, without having names for the kind of sequence we are working with, whether it is of Position or Step-size, we need to give names to the two operators\(^3\) we use for producing a sequences from sequences. So, if we start with any sequence,

\[ \{a_i\}_{i=1}^n = a_1, a_2, a_3, a_4, a_5, a_6, \ldots, a_{n-1}, a_n \]

\(^3\) In mathematics, we have “operations” like addition, exponentiation, etc, for producing numbers from numbers. We reserve the term operator for a thing we do to functions, as opposed to numbers. Thus, differentiation and integration are operators on functions. You apply differentiation to a function and
\[
\{a_i\}_{i=1}^n = a_1, a_2, a_3, a_4, a_5, a_6, \ldots, a_{n-1}, a_n
\]
then one thing we do to it is to form what we will call, its *Successive Difference*, which we denote by \(SD\).
\[
SD\left(\{a_i\}_{i=1}^n\right) = a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots, a_k - a_{k-1}, \ldots, a_n - a_{n-1}
\]
And we also form what we call its *Accumulation*, which we denote by \(ACC\).
\[
ACC\left(\{a_i\}_{i=1}^n\right) = a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, \ldots, \sum_{i=1}^k a_i, \ldots, \sum_{i=1}^n a_i
\]

As a way of focusing your report, imagine that you have come upon a high school in a very, very remote town called *Sequenceville*. All of the best math students in Sequenceville High take calculus, but the course they take in calculus is only about sequences. They are experts at applying the Successive Difference operator to a sequence and getting another sequence, and doing the same with the Accumulation operator. But they have no idea what they are doing, how the two concepts are related, and how any of this is connected to anything in the real world. In other words, they are not a whole lot different from many high school calculus students elsewhere who are taking a traditional calculus course. It’s just that the students in Sequenceville High School are confused about the calculus of sequences instead formulas. (At least they don’t have to worry about limits.)

You have been called in to help them understand what this subject calculus of sequences is all about. You have decided to do this by going carefully through the whole development of the ideas of calculus through the five Basic Tasks. In order to guide and motivate them, you probably want to link what you are saying about sequences to motion. But this is only for motivation. The development of the subject needs to be in terms of naked sequences.

Among your goals is to define two operators on graphs that are made up of a finite number of points. One operator, that we’ll call the \(GSD\) operator (for the graphical \(SD\) operator, until we get a better name), corresponds to the \(SD\) operator on sequences. The other is an operator on graphs, that we’ll call the \(GACC\) operator (for the graphical \(ACC\) operator) that corresponds to the \(ACC\) operator on sequences. But, for instance, when you get to Basic Task #2 and you are describing how to connect the shape of a graph with the shape of its \(GSD\) graph, then you cannot give an explanation in terms of speed or motion. It has to be in terms of the operation on graphs that you have defined. Among the most difficult tasks will be Basic Task #4, about the Reverse operator. The following was fairly obvious in the world of the Walking Game: If we begin with a Position sequence and form its corresponding Step-size sequence, and then form the Position sequence from that sequence, then the sequence you arrive at is just a function. We use the term operator here, because a sequence is a kind of function; it is a whole thing in itself made up of numbers.
the Position sequence you started from.  *It's not even clear how to say this in this new naked context.*

Your report on this Investigation should be a description of what you would say to the calculus students of Sequenceville High School. It should be structured, of course, by the five Basic Tasks, carried out in a naked sequence context. This is likely to take 4–7 pages, in order to be coherent and complete.

While writing this Report is your individual responsibility, we should probably plan on spending class time working on the mathematical ideas.