INDE 411: Stochastic Models and Decision Analysis

Lab 4: The Monopoly® chain

The objective of the lab is to let you experiment with Excel to model and analyze Markov chains. The lab starts with a generic introduction, and then lets you test your skills on the 'Monopoly Markov Chain'.

Part 0: Get ready

Open Microsoft Excel, click the Microsoft Office Button, and then click Excel Options. Click the add-ins category; in the manage box, click Excel add-ins, and then click Go. … Check ‘Solver Add-in’ if not already checked. Verify that the add-in is installed by checking if “Add-Ins” tab contains “Solver”.
Part 1: Basic Skills - Matrices (Arrays) in Excel

In Excel, it is usually straightforward to use formulas that refer to single cells. However multiplying, inverting, or taking powers of matrices is hard if you try to calculate each matrix element separately. A set of formulas called ‘array formulas’ make this easier, but there are very small tricks.

Type in two 4x4 matrices in Excel. For example:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Matrix 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>7</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>6</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Multiply the two matrices using the MMULT formula. The trick here is to select the area where the result would show first, then typing in the formula, then pressing \texttt{Ctrl+Shift+Enter}.

There are many array formulas in Excel. The ones you will find most useful in this class are MMULT and MINVERSE (for taking the inverse of a square array).

Try taking the inverse of one of your matrices. Also try to take the 6\textsuperscript{th} power of that array.
Part 2: A simple Markov chain – Steady State Probabilities in Excel

We will start with modeling a very simple Markov chain first. Consider its transition diagram below:

\[
P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.75 & 0.05 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}
\]

Start with a brand new Excel sheet. Enter your P matrix as you would just write it. A format like below is usually helpful:

Fill in the matrix. Then use your Excel skills to find the probability of moving from state A to state C in 3 transitions.

Is this Markov chain irreducible and ergodic (finite number of states in a single class, all aperiodic and recurrent)?

Solve for steady state probabilities in Excel
As you (hopefully) answered the previous question ‘yes’, we can analyze the steady-state behavior of this Markov chain.

Let’s try finding the steady-state probabilities of being in each of the three states. For this, we have to solve the steady-state equations:

\[
(1) \sum_{j=0}^{M} \pi_j = 1
\]

\[
(2) \pi_j = \sum_{i=0}^{M} \pi_i p_{ij} \quad \text{for all} \ j
\]

First, get some area ready for portions of these equations. The following format is useful:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Single-step transition matrix (P)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>3</td>
<td>0.7</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>4</td>
<td>B</td>
<td>0.2</td>
<td>0.75</td>
</tr>
<tr>
<td>5</td>
<td>C</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

We will make Excel find values of \( \Pi^T \) (cells B10..D10), so we don’t want to write any formulas there.

We have to write an array formula for \( \Pi^T P \), though, in cells B8..D8. Try it:

Note that \( \Pi^T e \) (in F10) is just the sum of B10..D10 (\( \pi \) values).
So all we have to do now is to tell Excel to find the \( \pi \) values such that equations (1) and (2) hold. Go to Add-Ins -> Solver. We want to tell Excel to set the sum of \( \pi \) values equal to 1, by changing those values, such that \( \pi^T P = \pi^T \).

![Excel Solver interface](image)

The solver is almost ready to find a solution to the equations now. But first, click on ‘Options’ and make sure ‘Assume Linear Model’ is checked:

![Solver Options interface](image)

Hit OK, then ‘Solve’.
Part 3: Pick your token

Part 4: Initial thoughts

We can model the Monopoly game as a Markov chain. Remember that discussion.

What simplifying assumptions should we make?

How could we characterize states?

Can the states just be our token location?

Or do we need any other information?

1) Assume we are at “GO” now. After a dice roll, what is the probability that we’ll land on Oriental Avenue? Other squares?

2) What is the effect of ‘Chance’ and ‘Community Chest’ squares?
Part 5: The Monopoly chain

Here is a classic American edition Monopoly board:

3) Define your state space. Be careful about the 'Jail' square. We need more than one state to represent it.

\[ X_t \in \{ 1: \text{Go}, 2: \text{Mediterranean, etc,} \, 3, 4, \ldots, 28, 29, 30, \text{Just Visiting/Jail}, \]
\[ J_1: \text{In jail for one turn,} \, J_2: \text{In jail for two turns,} \]
\[ 31, 32, \ldots, 40 \} \]

4) Define the one-step transition probabilities. Ignore the 'Chance' and 'Community Chest' cards.
that make you move. A transition matrix ready to be filled will be available on the course website. **Save a copy to your desktop before working on it.**

Next, find the steady-state probabilities. Use your learning in Part 2 to define necessary cells and setup Solver to solve the steady-state equations.

What do the results tell you?

If you could own any one property, which one would it be?

If you could own any one monopoly (set of properties of the same color), which one would it be?

Are these results enough, or do you need to incorporate other information?

Let’s practice with the following questions:

a. What is the fraction of time spent in jail?

b. What is the expected recurrence time for paying the “luxury tax”? Assume that during the game, money is only made and lost from rents. Ignore the “Chance” and “Community Chest” cards that make you move, and building houses or hotels.

c. If you could own any one of the 8 monopolies for free, which one would you pick to guarantee maximum (expected) rent income per turn? (rents of different properties are given in next page)

d. Now, assume that you have to pay the price of every property in a monopoly to own that monopoly (Prices are given in the table). Using your calculations from part (c), calculate the break-even point (expected number of turns before starting to make profit) for each monopoly. Which monopoly reaches the break-even point fastest?

<table>
<thead>
<tr>
<th>Property Name</th>
<th>Monopoly</th>
<th>Position</th>
<th>Price</th>
<th>Rent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mediterranean Ave.</td>
<td>Purple</td>
<td>2</td>
<td>60</td>
<td>2</td>
</tr>
<tr>
<td>Baltic Ave.</td>
<td>Purple</td>
<td>4</td>
<td>60</td>
<td>4</td>
</tr>
<tr>
<td>Oriental Ave.</td>
<td>Light-Blue</td>
<td>7</td>
<td>100</td>
<td>6</td>
</tr>
</tbody>
</table>