Markov Chains
(Part 3)

State Classification
State Classification
Accessibility

• State \( j \) is accessible from state \( i \) if \( p_{ij}^{(n)} > 0 \) for some \( n \geq 0 \), meaning that starting at state \( i \), there is a positive probability of transitioning to state \( j \) in some number of steps.

• This is written \( j \leftarrow i \)

• State \( j \) is accessible from state \( i \neq j \) if and only if there is a directed path from \( i \) to \( j \) in the state transition diagram.

• Note that every state is accessible from itself because we allow \( n = 0 \) in the above definition and \( p_{ii}^{(0)} = P(X_0 = i \mid X_0 = i) = 1 > 0 \).
State Classification Example 1

• Consider the Markov chain

\[
P = \begin{bmatrix}
0.4 & 0.6 & 0 & 0 & 0 \\
0.5 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0.3 & 0.7 & 0 \\
0 & 0 & 0.5 & 0.4 & 0.1 \\
0 & 0 & 0 & 0.8 & 0.2
\end{bmatrix}
\]

• Draw its state transition diagram

• Which states are accessible from state 0?
  States 0 and 1 are accessible from state 0

• Which states are accessible from state 3?
  States 2, 3, and 4 are accessible from state 3

• Is state 0 accessible from state 4?
  No
State Classification Example 2

- Now consider a Markov chain with the following state transition diagram

- Is state 2 accessible from state 0? Yes
- Is state 0 accessible from state 2? No
- Is state 1 accessible from state 0? Yes
- Is state 0 accessible from state 1? No
State Classification

Communicability

• States $i$ and $j$ communicate if state $j$ is accessible from state $i$, and state $i$ is accessible from state $j$ (denote $j \leftrightarrow i$)

• Communicability is
  – **Reflexive:** Any state communicates with itself
  – **Symmetric:** If state $i$ communicates with state $j$, then state $j$ communicates with state $i$
  – **Transitive:** If state $i$ communicates with state $j$, and state $j$ communicates with state $k$, then state $i$ communicates with state $k$

• For the examples, which states communicate with each other?
State Classification
Communicability

• Example 1:

0.6
0.4
0.5
0.5
0.3
0.7
0.1
0.2

0 ↔ 1, 2 ↔ 3 ↔ 4

• Example 2:

0 ↔ 0, 1 ↔ 1, 2 ↔ 2

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State Classes

- Two states are said to be in the same class if the two states communicate with each other, that is \( i \leftrightarrow j \), then \( i \) and \( j \) are in same class.

- Thus, all states in a Markov chain can be partitioned into disjoint classes
  - If states \( i \) and \( j \) are in the same class, then \( i \leftrightarrow j \).
  - If a state \( i \) is in one class and state \( j \) is in another class, then \( i \) and \( j \) do not communicate.

- How many classes exist in the examples?
- Which states belong to each class?
State Classes

- Example 1:

- Example 2:
Gambler’s Ruin Example

- Consider the gambling game with probability $p=0.4$ of winning on any turn

- State transition diagram and one-step transition probability matrix:

- How many classes are there?
  Three: $\{0\}$ $\{1,2\}$ $\{3\}$
Irreducibility

• A Markov chain is irreducible if all states belong to one class (all states communicate with each other).
• If there exists some \( n \) for which \( p_{ij}^{(n)} > 0 \) for all \( i \) and \( j \), then all states communicate and the Markov chain is irreducible.
• If a Markov chain is not irreducible, it is called reducible.
• If a Markov chain has more than one class, it is reducible.
• Are the examples reducible or irreducible?
  
  Ex 1: Reducible  \{0,1\} \{2,3,4\}
  Ex 2: Reducible  \{0\} \{1\} \{2\}
  Gambler’s Ruin Ex: Reducible  \{0\} \{1,2\} \{3\}
Examples of Irreducible Chains

- **Weather example**

  - Transition diagram:
    - Sun (0) with transition probabilities:
      - Stay in Sun with probability $p$.
      - Move to Rain with probability $q$.
      - Move to Sun with probability $1-p$.
      - Move to Rain with probability $1-q$.

- **Inventory example**

  - Transition diagram:
    - States: $0, 1, 2, 3$ (represent inventory levels).
    - Transitions:
      - From $0$ to $1$, $2$, or $3$ with probabilities $P(D=1)$, $P(D=2)$, $P(D=3)$.
      - From $1$ to $0$, $1$, or $2$ with probabilities $P(D=0)$, $P(D=1)$, $P(D=2)$.
      - From $2$ to $0$, $1$, or $2$ with probabilities $P(D=0)$, $P(D=1)$, $P(D=2)$.
      - From $3$ to $0$ with probability $P(D=0)$.
Periodicity of the Gambler’s Ruin

- Observe: if you start in State 1 at time 0, you can only come back to it in times 2, 4, 6, 8, …
- In other words, 2 is the greatest common denominator of all integers \( n > 0 \), for which \( p_{ii}^{(n)} > 0 \)
- We say, the period of State 1 is 2. The period of State 2 is also 2. And observe, they are in the same class.
- State 0 has a period of 1, called aperiodic
Periodicity

• The **period** of a state $i$ is the greatest common denominator (gcd) of all integers $n > 0$, for which $p_{ii}^{(n)} > 0$

• Periodicity is a *class property*
  – If states $i$ and $j$ are in the same class, then their periods are the same

• State $i$ is called **aperiodic** if there are two consecutive numbers $s$ and $(s+1)$ such that the process can be in state $i$ at these times, i.e., the period is 1
Periodicity Examples

• Which of the following Markov chains are aperiodic?
• Which are irreducible?

\[
P = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

Irreducible, because all states communicate
Period = 3

\[
P = \begin{bmatrix}
\frac{1}{3} & \frac{2}{3} & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} \\
0 & \frac{1}{4} & \frac{3}{4}
\end{bmatrix}
\]

Irreducible
Aperiodic

\[
P = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & \frac{1}{4} & \frac{3}{4}
\end{bmatrix}
\]

Reducible, 2 classes
Each class is aperiodic

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Transient States

- Consider the gambling example again:
  - Suppose you are in state 1. What is the probability that you will never return to state 1 again?
  - For example, if you win in state 1, and then win again in state 2, then you will never return to state 1 again. The probability this happens in $0.4 \times 0.4 = 0.16$
  - Thus, there is a positive probability that, starting in state 1, you will never return to state 1 again.
  - State 1 is called a transient state.

- In general, a state $i$ is said to be transient if, upon entering state $i$, there is a positive probability that the process may never return to state $i$ again
- A state $i$ is transient if and only if there exists a state $j$ (different from $i$) that is accessible from state $i$, but $i$ is not accessible from $j$
- In a finite-state Markov chain, a transient state is visited only a finite number of times
Recurrent States

- A state that is not transient is called recurrent.
- State \( i \) is said to be recurrent if, upon entering state \( i \), the process will definitely return to state \( i \).
- Since a recurrent state definitely will be revisited after each visit, it will be visited infinitely often.
- A special type of recurrent state is an absorbing state, where, upon entering this state, the process will never leave it. State \( i \) is an absorbing state if and only if \( p_{ii} = 1 \).
- Recurrence (and transience) is a class property.
- In a finite-state Markov chain, not all states can be transient:
  - Why? Because there has to be another state \( j \) to move to, so there would have to be \( \infty \) states.
Transient and Recurrent States
Examples

• **Gambler’s ruin:**
  – Transient states: \{1, 2\}
  – Recurrent states: \{0\} \{3\}
  – Absorbing states: \{0\} \{3\}

• **Inventory problem**
  – Transient states: None
  – Recurrent states: \{0, 1, 2, 3\}
  – Absorbing states: None
Examples of Transient and Recurrent States

- Transient \{0\} \{1\}, Period = 2
- Recurrent \{2,3\}, Period = 2
- Transient \{0\} \{1\} Aperiodic
- Absorbing \{2\} Aperiodic
- Recurrent \{0,1,2\}
- Period = 3
- Recurrent \{0,1\}, Aperiodic
Ergodic Markov Chains

• In a finite-state Markov chain, not all states can be transient, so if there are transient states, the chain is reducible
• If a finite-state Markov chain is irreducible, all states must be recurrent
• In a finite-state Markov chain, a state that is recurrent and aperiodic is called ergodic
• A Markov chain is called ergodic if all its states are ergodic.
• We are interested in irreducible, ergodic Markov chains...