Homework #1 Solutions

1.1
1.9
1.17
1.11
1.15
1.26
2.12
2.29
2.54
Let $x_{ij}$ be the amount of ingredient $i$, used in field $j$, where $i = C, L, S, F$ represents corn, limestone, soybean, fishmeal, and $j = 1, 2, 3$ represent cattle, sheep, and chicken, respectively.

Minimize $Z = 0.20 \sum x_{i1} + 0.12 \sum x_{i2} + 0.24 \sum x_{i3} + 0.12 \sum x_{i4}$

subject to:

$\frac{3}{2} x_{c1} \leq 6$ $\frac{3}{2} x_{s1} \leq 4$

$\frac{3}{2} x_{c2} \leq 10$ $\frac{3}{2} x_{s2} \leq 5$

$\sum_{i} x_{c1} \geq 10$ $\sum_{i} x_{c2} \geq 6$ $\sum_{i} x_{c3} \geq 8$

$8x_{c1} + 6x_{c2} + 10x_{s1} + 4x_{f1} \geq 6(x_{c1} + x_{c2} + x_{s1} + x_{f1})$

$8x_{c2} + 6x_{c2} + 10x_{s2} + 4x_{f2} \geq 6(x_{c2} + x_{c2} + x_{s2} + x_{f2})$

$6(x_{c3} + x_{s3} + x_{f3}) \geq 6(x_{c3} + x_{c3} + x_{s3} + x_{f3})$

$10x_{c1} + 5x_{c2} + 12x_{s1} + 8x_{f1} \geq 6(x_{c1} + x_{c2} + x_{s1} + x_{f1})$

$10x_{c2} + 5x_{c2} + 12x_{s2} + 8x_{f2} \geq 6(x_{c2} + x_{c2} + x_{s2} + x_{f2})$

$10x_{c3} + 5x_{c3} + 12x_{s3} + 8x_{f3} \geq 6(x_{c3} + x_{c3} + x_{s3} + x_{f3})$

$6x_{c1} + 10x_{c2} + 6x_{s1} + 6x_{f1} \geq 7(x_{c1} + x_{c2} + x_{s1} + x_{f1})$

$6x_{c2} + 10x_{c2} + 6x_{s2} + 6x_{f2} \geq 6(x_{c2} + x_{c2} + x_{s2} + x_{f2})$

$6x_{c3} + 10x_{c3} + 6x_{s3} + 6x_{f3} \geq 6(x_{c3} + x_{c3} + x_{s3} + x_{f3})$

$4(x_{c1} + x_{c1} + x_{s1} + x_{f1}) \leq 8x_{c1} + 6x_{c2} + 6x_{s1} + 7x_{f1} \leq 8(x_{c1} + x_{c1} + x_{s1} + x_{f1})$

$4(x_{c2} + x_{c2} + x_{s1} + x_{f2}) \leq 8x_{c2} + 6x_{c2} + 6x_{s2} + 4x_{f2} \leq 6(x_{c2} + x_{c2} + x_{s2} + x_{f2})$

$4(x_{c3} + x_{c3} + x_{s3} + x_{f3}) \leq 8x_{c3} + 6x_{c3} + 6x_{s3} + 9x_{f3} \leq 6(x_{c3} + x_{c3} + x_{s3} + x_{f3})$
launching a rocket to a fixed altitude $b$ in a given time $T$

Let $u(t)$ be the acceleration force exerted at time $t$,
and $y(t)$ the rocket altitude at time $t$. We have

$$\min \int_0^T |u(t)| dt$$

subject to

$$y''(t) = u(t) - g$$

$$y(T) = b$$

$$y(t) = 0 \quad t \in [0, T]$$

Discute the above formulation.

let $n$ be the # of subintervals, of width $\Delta = \frac{T}{n}$

let $u_j = u(t_j) = u_j(j\Delta)$

let $y_j = y(t_j) = y_j(j\Delta)$

let $x_j = |u_j|$

Notice that a finite difference approx yields:

$$\frac{y_j(t_{j+1}) - y_j(t_j)}{\Delta} = \frac{\dot{y}(t_j + \Delta) - \dot{y}(t_j)}{\Delta}$$

$$= \frac{y(t_j + \Delta) - y(t_j)}{\Delta}$$

$$= \frac{y(t_j + \Delta) - 2y_j + y_j - \Delta}{\Delta^2}$$

New formulation:

$$\min \sum_{j=0}^{n-1} x_j (\Delta)$$

subject to

$$y_{j+1} - 2y_j + y_{j-1} = \Delta^2 (u_j - g) \quad (j=1, \ldots, n-1)$$

$$y_0 = b$$

$$y_j \geq 0$$

$$x_j \geq u_j$$

$$x_j \geq u_j$$
This works for \( x_j = |u_j| \) because \( x_j \geq u_j \) and \( x_j \geq -u_j \) forces \( x_j \) to be either one or the other.

Now the formulation has \( u_j \) unrestricted in sign.

Another way is:

Let \( u_j = x_j^+ - x_j^- \)

Then \( |u_j| = x_j^+ + x_j^- \)

and we need \((x_j^+(x_j^-)) = 0\)

Substituting throughout, we get,

\[
\min_{j=0}^{n-1} \sum_{j=0}^{n-1} x_j^+ + x_j^-
\]

s.t.

\[
y_j = 2y_j - y_{j-1} - \Delta^2 (x_j^+ - x_j^- - g) \text{ for } j=1, \ldots, n-1
\]

\[
y_n = b
\]

\[
y_0 = 0
\]

\[
y_j \geq 0 \quad j=0,1,\ldots,n-1
\]

\[
x_j^+ \geq 0
\]

\[
x_j^- \geq 0
\]

AND \((x_j^+(x_j^-)) = 0\) but this is satisfied by properties of BFS.
$X_{ij} = \text{amount of money collected at beginning of year } i, \text{ in } j\text{-year time deposits}$

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
<th>Year 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{11}$</td>
<td>$X_{12}$</td>
<td>$X_{13}$</td>
<td>$X_{14}$</td>
<td>$X_{15}$</td>
</tr>
<tr>
<td>$1 - X_{11}$</td>
<td>$1 - X_{12}$</td>
<td>$1 - X_{13}$</td>
<td>$1 - X_{14}$</td>
<td>$1 - X_{15}$</td>
</tr>
</tbody>
</table>

$1 - X_{21}$

$1 - X_{22}$

$1 - X_{23}$

$1 - X_{31}$

$1 - X_{32}$

$1 - X_{33}$

$1 - X_{41}$

$1 - X_{42}$

$1 - X_{51}$

Max (and at end of year 5)

$1.27X_{33} + 1.17X_{42} + 1.08X_{51}$

s.t.

\[
\begin{align*}
X_{11} + X_{12} &\leq 2200 \\
X_{21} + X_{22} + X_{23} &\leq 1.08X_{11} + (2200 - X_{11} - X_{12}) \\
X_{31} + X_{32} + X_{33} &\leq 1.17X_{12} + 1.08X_{21} \\
X_{41} + X_{42} &\leq 1.17X_{22} + 1.08X_{31} \\
X_{51} &\leq 1.27X_{33} + 1.17X_{42} + 1.08X_{41}
\end{align*}
\]

Or, because it is always profitable to invest, we could use (=)

$X_{11} + X_{12} = 2200$

$X_{21} + X_{22} + X_{23} = 1.08X_{11}$

$X_{31} + X_{32} + X_{33} = 1.17X_{12} + 1.08X_{21}$

$X_{41} + X_{42} = 1.17X_{22} + 1.08X_{31}$

$X_{51} = 1.27X_{33} + 1.17X_{42} + 1.08X_{41}$

$X_{11,1}, X_{12,1}, X_{21,1}, X_{22,1}, X_{23,1}, X_{31,1}, X_{32,1}, X_{33,1}, X_{41,1}, X_{42,1}, X_{51,1} \geq 0$
Let $X_1 =$ # of barrels of light crude oil, $X_2 =$ # of heavy.

\[
\begin{align*}
\text{min} & \quad 11X_1 + 9X_2 \\
n & \quad 0.4X_1 + 0.32X_2 \geq 1,000,000 \\
& \quad 0.2X_1 + 0.4X_2 \geq 400,000 \\
& \quad 0.35X_1 + 0.2X_2 \geq 250,000 \\
& \quad X_1, X_2 \geq 0
\end{align*}
\]

Let $x_{ij}$ be the # of units of product $i$ manufactured on machine $j$, $i = 1, 2, 3$, $j = 1, 2, 3, 4$

\[
\begin{align*}
\text{min} & \quad (4x_{11} + 4x_{12} + 5x_{13} + 7x_{14}) \\
& \quad + (6x_{21} + 7x_{22} + 6x_{23} + 6x_{24}) \\
& \quad + (12x_{31} + 10x_{32} + 8x_{33} + 11x_{34}) \\
& \quad \text{cost}
\end{align*}
\]

\[
\begin{align*}
\text{st.} & \quad x_{11} + x_{12} + x_{13} + x_{14} \geq 4000 \\
& \quad x_{21} + x_{22} + x_{23} + x_{24} \geq 5000 \\
& \quad x_{31} + x_{32} + x_{33} + x_{34} \geq 3000
\end{align*}
\]

\[
\begin{align*}
& \quad 0.3x_{11} + 0.2x_{21} + 0.8x_{31} \leq 1500 \\
& \quad 0.25x_{12} + 0.3x_{22} + 0.6x_{32} \leq 1200 \\
& \quad 0.2x_{13} + 0.2x_{23} + 0.6x_{33} \leq 1500 \\
& \quad 0.2x_{14} + 0.25x_{24} + 0.5x_{34} \leq 2000
\end{align*}
\]

\[
\begin{align*}
& \quad x_{ij} \geq 0 \quad (i = 1, 2, 3, \quad j = 1, 2, 3, 4)
\end{align*}
\]
(1) If c is same direction as c₁, then all points on half-line l₁ are optimal.
(2) If c lies between c₁ and c₂, then a is optimal.
(3) If c is same direction as c₂, then all points on ab are optimal.
(4) If c lies between c₂ and c₃, then b is optimal.
(5) If c is same direction as c₃, then all points on bc are optimal.
(6) If c lies between c₃ and c₄, then c is optimal.
(7) If c is same direction as c₄, then all points on half-line l₂ are optimal.
(8) If c lies between c₄ and c₁, then no optimum exists (unbounded).
2.12 Show that if $A \neq B$ are non-singular matrices that are both invertible, then $(AB)^{-1} = B^{-1}A^{-1}$.

We know that $AA^{-1} = I$ and $BB^{-1} = I$ because both $A \neq B$ are invertible.

Check that $(AB)(AB)^{-1} = I$:

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$.

Also $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$.

So $(AB)^{-1} = B^{-1}A^{-1}$.

2.29

a) $\mathcal{E}(x_1, x_2): x_1^2 + x_2^2 \geq 1$

Not convex

$(x_1, x_2) = (0, 0)$

$(1, 1) \in \mathcal{E}$

$(-1, -1) \in \mathcal{E}$

$(0, 0) = \frac{1}{2}(1, 1) + \frac{1}{2}(-1, -1)$ but $(0, 0) \notin \mathcal{E}$

b) $\{(x_1, x_2, x_3): x_1 + x_2 \leq 1, x_1 - x_2 \leq 2\}$

Yes, convex - Intersection of half-spaces

(c) $\mathcal{E}(x_1, x_2): x_2 - x_1 = 0$

Not convex

$(1, 1) \in \mathcal{E}$

$(-1, -1) \in \mathcal{E}$

$(0, 1) = \frac{1}{2}(1, 1) + \frac{1}{2}(-1, -1)$ but $(0, 1) \notin \mathcal{E}$

d) $\{(x_1, x_2, x_3): x_2 \geq x_1^2, x_1 + x_2 + x_3 \leq 6\}$

Yes, convex - finite intersection of convex sets

e) $\mathcal{E}(x_1, x_2): x_1 = 1, |x_2| \leq 4$

Yes, convex - finite intersection of convex sets

f) $\mathcal{E}(x_1, x_2, x_3): x_3 = |x_2|, x_1 \leq 4$

Not convex

$\mathcal{E}(x_2, x_3): x_2 = 1, x_3 = 1$.

$\forall \mathcal{E}(x_1, x_2, x_3) = \frac{1}{2}(1, 1) + \frac{1}{2}(-1, -1)$
2.54

a) Is it possible for \( \mathcal{I} = \{ x \mid Ax \leq b, x \geq 0^3 \} \) to be empty, but \( D = \{ d \mid Ad \leq 0, 1d = 1, d \geq 0^3 \} \) to be nonempty?

Yes, consider \( \mathcal{I} = \{ x_i, x_2 \} : x_1 - x_2 = 0, x_1 - x_2 = 1, x_1, x_2 \geq 0 \)

\[ \mathcal{I} \text{ is empty} \]

\[ D = \{ d_1, d_2 \} : d_1 - d_2 = 0, d_1 + d_2 = 1, d_1, d_2 \geq 0 \]

\[ = \{ \frac{1}{2}, \frac{1}{2} \} \]

b) Is there a relationship between redundancy and degeneracy of a polyhedral set?

A redundant constraint may cause degeneracy but only if it "saves the polyhedron I".

Degeneracy may or may not be resolved by eliminating redundant constraints.

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c) Does degeneracy imply redundancy in 2-dims?

Yes

d) If the intersection of a finite number of half-spaces is non-empty, then this set has at least one extreme point. True or False?

False

\[ x_2 \leq 1 \]

\[ x_2 < -1 \]
e) An unbounded n-dim. polyhedral set can have at most n extreme points. True or False?

False

f) What is the maximum (actual) dimension of \( \mathcal{X} = \{ x : A x = b, x \geq 0 \} \) where \( A \) is \( m \times n \), of rank \( t \) and \( t \leq m \leq n \)?

\( n - t \)

On page 64, "Now given any face \( F \) of \( \mathcal{X} \), if \( r(F) \) is the maximum number of linearly independent defining hyperplanes binding at all points feasible to \( \mathcal{X} \), then the claim of \( F \) is: \( \dim(F) = n - r(F) \)."

"Similarly, if \( r(\mathcal{X}) \) is defined w.r.t. \( \mathcal{X} \) itself, then although \( \mathcal{X} \subseteq \mathcal{F} \), it is actually of dimension \( \dim \mathcal{X} = n - r(\mathcal{X}) \)."