Problem 6.2

(P)
Min \( 2x_1 + 15x_2 + 5x_3 + 6x_4 \)
ST
\[
\begin{align*}
x_1 + 6x_2 + 3x_3 + x_4 & \geq 2 \quad \Leftrightarrow w_1 \\
-2x_1 + 5x_2 - x_3 + 3x_4 & \leq -3 \quad \Leftrightarrow w_2 \\
2x_1 - 5x_2 + x_3 - 3x_4 & \geq 3 \\
x_1, x_2, x_3, x_4 & \geq 0
\end{align*}
\]

(-1) \* \( (-2x_1 + 5x_2 - x_3 + 3x_4 \leq -3) \quad \Leftrightarrow w_2 \\
(2x_1 - 5x_2 + x_3 - 3x_4 \geq 3) \quad w_1, w_2 \geq 0

a) (D)
Max \( 2w_1 + 3w_2 \)
ST
\[
\begin{align*}
w_1 + 2w_2 & \leq 2 \quad \Leftrightarrow x_1 \\
6w_1 - 5w_2 & \leq 15 \quad \Leftrightarrow x_2 \\
3w_1 + w_2 & \leq 5 \quad \Leftrightarrow x_3 \\
w_1 - 3w_2 & \leq 6 \quad \Leftrightarrow x_4 \\
w_1, w_2 & \geq 0
\end{align*}
\]
b)
\[ z^* = 3 \frac{4}{5} \]
Constrain 2 is not binding, so \( x_2^* = 0 \)
Constrain 2 is not binding, so \( x_2^* = 0 \)
\( w_1^*, w_2^* > 0 \), so primal two constraints must be binding \( x_5^*, x_6^* = 0 \)
we have
\[
\begin{align*}
x_1^* + 3x_3^* & = 2 \\
2x_1^* + x_3^* & = 3 \\
x_1^* & = \frac{8}{5}, \ x_3^* = \frac{1}{5}
\end{align*}
\]

Problem 6.5

(P) Min \( c_1x_1 + c_2x_2 + c_3x_3 \)
ST
\[
\begin{align*}
A_{11}x_1 + A_{12}x_2 + A_{13}x_3 & \geq b_1 \quad \Leftrightarrow w_1 \\
(A_{21}x_1 + A_{22}x_2 + A_{23}x_3 & \geq b_2) *(-1) \quad \Leftrightarrow w_2 \\
A_{31}x_1 + A_{32}x_2 + A_{33}x_3 & \geq b_3 \quad \Leftrightarrow w_3 \\
x_1, x_2, x_3 & \geq 0, \ x_2 \leq 0, \ x_3 \text{ unrestricted}
\end{align*}
\]

(D)
Max \( b_1w_1 + b_2w_2 + b_3w_3 \)
\[
\begin{align*}
A_{11}w_1 + A_{21}w_2 + A_{31}w_3 & \geq c_1 \quad \Leftrightarrow x_1 \\
A_{12}w_1 + A_{22}w_2 + A_{32}w_3 & \geq c_2 \quad \Leftrightarrow x_2 \\
A_{13}w_1 + A_{23}w_2 + A_{33}w_3 & \geq c_3 \quad \Leftrightarrow x_3 \\
w_1, w_2, w_3 & \geq 0, \ w_2 \leq 0, \ w_3 \text{ unrestricted}
\end{align*}
\]
Problem 6.7

(P) \hspace{1cm} (D)

Min \ cx \hspace{1cm} Max \ wb

ST \ Ax \geq b \hspace{1cm} ST \ wA \leq c

x \geq 0 \hspace{1cm} w \geq 0

a) False – both could be infeasible

b) for the LP:
Min \ x_1
ST \ 2x_1 - x_2 \geq 0 \iff w_1
-2x_1 + 3x_2 \geq -6 \iff w_2

(D)

Max -6w_2
ST \ 2w_1 - 2w_2 \leq 1 \iff x_1
-w_1 + 3w_2 \leq 0 \iff x_2
w_1, w_2 \geq 0

Consider \ \mathbf{x_B} = \begin{bmatrix} x_i \\ x_4^s \end{bmatrix} \quad \text{Give complementary solution}

x = (0, 0, 0, 6)
if \ x_4^s \ is \ basic \ w_2 = 0
if \ x_1 \ is \ basic, \ then \ w_3^s = 0
so \ 2w_1 = 1 \Rightarrow w_1 = \frac{1}{2} \ and \ then \ w_4^s = \frac{1}{2}
\ w = (0, \frac{1}{2}, 0, \frac{1}{2})

Both complementary solutions are feasible
Basis for (P) is degenerate, for (D) is nondegenerate.

c) True – if (P) has alternate optimal, (D) is degenerate and vice versa.

d) Yes – Dual solution could be degenerate

e) False – (P) unbounded \Rightarrow (D) infeasible and changing rhs in (P) is changing c in (D), which won’t affect dual feasibility.

f) if \ \mathbf{x_B} = (x_2, x_4, x_5) \ then \ \mathbf{x_N} = (x_1, x_3)

primal basic solution is infeasible (and degenerate?)
Dual basic solution is feasible, optimal and degenerate
**Problem 6.9**

(P)

Min \(2x_1 + 3x_2 + 6x_3\)

ST

\[
\begin{align*}
  x_1 + 2x_2 + 3x_3 & \leq 10 \quad (+x_4^S) \quad \Leftrightarrow \quad w_1 \\
  x_1 - 2x_2 + 2x_3 & \leq 6 \quad (+x_5^S) \quad \Leftrightarrow \quad w_2 \\
  x_1, x_2, x_3, x_4^S, x_5^S & \geq 0
\end{align*}
\]

a) (D)

Max \(10w_1 + 6w_2\)

ST

\[
\begin{align*}
  w_1 + w_2 & \geq 2 \quad (+w_3^S) \quad \Leftrightarrow \quad x_1 \\
  2w_1 - 2w_2 & \geq 3 \quad (+w_4^S) \quad \Leftrightarrow \quad x_2 \\
  3w_1 + 2w_2 & \geq 6 \quad (+w_5^S) \quad \Leftrightarrow \quad x_3 \\
  w_1, w_2, w_3^S, w_4^S, w_5^S & \geq 0
\end{align*}
\]

b)

\[
\begin{align*}
w_1 &= 0, \quad w_2 = 0, \quad z = 0, \quad wb = 0 \\
w_3 &= -2, \quad w_4 = -3, \quad w_5 = -6
\end{align*}
\]

complementary dual solution is infeasible

Let \(x_3\) enter and \(x_5^S\) leave

\[
\begin{align*}
w_1 &= 0, \quad w_2 = 3, \quad z = 18, \quad wb = 18 \\
w_3 &= 1, \quad w_4 = -9, \quad w_5 = 0
\end{align*}
\]

only one dual non-negative constraint is violated

Let \(x_2\) enter and \(x_4^S\) leave

\[
\begin{align*}
w_1 &= \frac{9}{5}, \quad w_2 = \frac{3}{10}, \quad z^* = \frac{99}{5} = wb \\
w_3 &= \frac{1}{10}, \quad w_4 = -9, \quad w_5 = 0
\end{align*}
\]

complementary dual solution is feasible
Problem 6.18

\[
\text{Min } \quad cx \\
\text{ST } \quad Ax = b \\
\quad l \leq x \leq u
\]

a) (P)

\[
\text{Min } \quad cx \\
\text{ST } \quad Ax = b \iff w_A \\
x \geq l \iff w_l \\
-x \geq -u \iff w_u \\
x \text{ is unrestricted in sign}
\]

(D)

\[
\text{Max } \quad w_Ab + w_l + w_uu \\
\text{ST } \quad w_Ab + w_l - w_u = c \\
w_A \text{ unrestricted in sign} \\
w_l \geq 0, w_u \geq 0
\]

b) the dual always possesses feasible solution.

For example.

let \( w_A = 0 \), then \( w_l - w_u = c \)

\[
w_l, w_u \geq 0
\]

if \( c \geq 0 \), let \( w_l = c \), and \( w_u = 0 \)

if \( c < 0 \), let \( w_u = 0 \), and \( w_u = |c| \)

another way to write this is:

\[
w_{lj} = \max (0, c_j), w_{uj} = \max (0, -c_j)
\]

c) If the primal possesses a feasible solution, and from (6) we know the dual possesses a feasible solution, then neither problem can be unbounded and so both problems have finite optimal solutions with equal objectives.
Problem 6.15

Two types of Chairs: 1- Top of the line; \( \frac{3}{2} \) hours assembly; 1 hour finishing
2 – Second line; \( \frac{1}{2} \) hours assembly; \( \frac{1}{2} \) hour finishing
Let \( x_i = \# \) of chairs of type \( i, i = 1,2 \)

(Primal)
Maximize \( Z = 20x_1 + 12x_2 \) (profit)
Subject to
\[
1.5x_1 + 0.5x_2 \leq 80 \quad \text{(Assembly)} \iff w_1
\]
\[
1.0x_1 + 0.5x_2 \leq 100 \quad \text{(finishing)} \iff w_2
\]
\[
x_1, x_2 \geq 0
\]

(Dual)
Minimize \( 100w_1 + 80w_2 \)
Subject to
\[
1.0w_1 + 1.0w_2 \geq 20
\]
\[
0.5w_1 + 0.5w_2 \geq 12
\]
\[
w_1, w_2 \geq 0
\]

Optimal Solution
\[
x_1 = 0
\]
\[
x_2 = 160
\]
\[
x_3 = 20
\]
\[
x_4 = 0
\]

with \( B = \begin{bmatrix} a_3 & a_2 \\ 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \)
\[
B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}
\]

\[
Z_1 - c_1 = c_BB^{-1}a_1 - c_1 = 0 \left[ \begin{array}{cc} 1 & -1 \\ 0 & 2 \end{array} \right] \begin{bmatrix} \frac{1}{2} \end{bmatrix} - 20 = -20
\]
\[
= 0 \left[ \begin{array}{cc} 0 & 24 \\ 0 & 1 \end{array} \right] \begin{bmatrix} 1 \end{bmatrix} - 20 = 4
\]

\[
Z_4 - c_4 = c_BB^{-1}a_4 - c_4
\]
\[
= 0 \left[ \begin{array}{cc} 0 & 24 \\ 0 & 1 \end{array} \right] \begin{bmatrix} 1 \end{bmatrix} - 0 = 24
\]

\( w_1 = 0; w_2 = 24; z = 1920 \)

At the optimal Solution;
\( w_1 = Z_3 - c_3 = 0 \) (since \( X_3 \) is basic)
so an additional hour in assembly will not increase profit (constrain is not binding)

\( w_2 = Z_4 - c_4 = 24 \)
so an additional hour in finishing is worth 24$

You could solve the dual problem directly to get \( w_1^* \) and \( w_2^* \)

Graph of Primal Constraints
So optimal solution is (0,24), so the marginal worth of assembly and finishing is (0,24).

**Problem 6.25**

Two person zero-sum game;
if row player selects strategy i and column player selects strategy j,
Row player gets $c_{ij}$ and column player gets $-c_{ij}$.

Let $x_i =$ probability the column player chooses strategy $i$, $i = 1,2,3$

a) **(Primal)**

Minimize $z$

Subject to

\[
\begin{align*}
x_1 + x_2 + x_3 &= 1 \quad \Leftrightarrow \quad w_0 \\
z - 2x_1 + x_2 &\geq 0 \quad \Leftrightarrow \quad w_1 \\
z + 3x_1 - x_2 - x_3 &\geq 0 \quad \Leftrightarrow \quad w_2 \\
x_1, x_2, x_3 &\geq 0 \\
z &\text{ unrestricted}
\end{align*}
\]

a) & b) **(Dual)**

Minimize $w_0$

Subject to

\[
\begin{align*}
w_0 + w_1 &= 1 \quad \Leftrightarrow \quad Z \\
w_0 - 2w_1 + 3w_2 &\leq 0 \quad \Leftrightarrow \quad x_1 \\
w_0 + w_1 - w_2 &\leq 0 \quad \Leftrightarrow \quad x_2 \\
w_0 - w_2 &\leq 0 \quad \Leftrightarrow \quad x_3 \\
w_0 &\text{ unrestricted} \\
w_0, w_2 &\geq 0
\end{align*}
\]

The row player’s problem:

$w_1, w_2 =$ probability the row player chooses strategy $i$, $i = 1,2$

$w_0 =$ expected payoff to the column player
c) since, \( w_2 = 1 - w_1 \)
So,
\[
\begin{align*}
  w_1 & \geq -\frac{1}{5}w_0 + \frac{3}{5} \\
  w_1 & \leq -\frac{1}{2}w_0 + \frac{1}{2} \\
  w_1 & \leq w_0 + 1 
\end{align*}
\]

Optimal Probabilities:
Expected payoff to column player:
\( w_0^* = -\frac{1}{7}, w_1^* = \frac{4}{7}, w_2^* = \frac{3}{7}, \)

\[ \text{Graph of Dual Constraints} \]

\[ \text{Graph of Primal Constraints} \]

d) and e)

Let \( x_4^S \) be surplus variable to constraint (2) in (P)
\( x_5^S \) be surplus variable to constraint (3) in (P)
\( x_3^S \) be slack variable to constraint (2) in (D)
\( x_4^S \) be slack variable to constraint (3) in (D)
\( x_5^S \) be slack variable to constraint (4) in (D)

\[
\begin{align*}
  x_1 + x_2 &= 1 \\
  z - 2x_1 + x_2 &= 0 \\
  z + 3x_1 - x_2 &= 0
\end{align*}
\]
\[
\Rightarrow \begin{cases}
  z - 2x_1 + (1 - x_1) = 0 \\
  z + 3x_1 + (1 - x_1) = 0
\end{cases}
\]

(Primal)
Minimize \( z \)
Subject to
\[
\begin{align*}
  x_1 + x_2 + x_3 &= 1 \iff w_0 \\
  z - 2x_1 + x_2 &\geq 0 \iff w_1 \\
  z + 3x_1 - x_2 - x_3 &\geq 0 \iff w_2 \\
  x_1, x_2, x_3 &\geq 0 \\
  w_0 &\text{ unrestricted}
\end{align*}
\]

(Dual)
Minimize \( w_0 \)
Subject to
\[
\begin{align*}
  w_1 + w_2 &= 1 \iff Z \\
  w_0 - 2w_1 + 3w_2 &\leq 0 (+w_3^S) \iff x_1 \\
  w_0 + w_1 - w_2 &\leq 0 (+w_4^S) \iff x_2 \\
  w_0 - w_2 &\leq 0 (+w_5^S) \iff x_3 \\
  w_0 &\text{ unrestricted} \\
  w_0, w_2 &\geq 0
\end{align*}
\]

Interpretation:
Since \( w_5^S \) is positive, then in constraint (3) in (D), \( w_0 \leq w_2 \) so when the new player tries to max the minimum expected payoff, \( 0w_1 - w_2 \) is not the solution, so the column player never chooses strategy 3, so \( x_3^* = 0 \). Note \( x_3 \) is associated with \( w_5^S \). Another way to see this is because strategy 2 eliminates strategy 3 for the column player (the column player gets \(-c_{ij}\) ) so strategy 3 will never get played.
Problem 6.29

a) \( \max c'x \quad \min w'b \)

ST \( Ax \leq b \)

\( x \geq 0 \)

ST \( wA \geq c \)

\( w \geq 0 \)

<table>
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<tr>
<th>Slacks</th>
<th>( Z' )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
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<th>( x_5^S )</th>
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<td>1</td>
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<td>0</td>
<td>-1</td>
<td>1</td>
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</table>

a) \( x_1^* = 2; \ x_2^* = 0; \ x_3^* = 1.5; \ x_4^* = 0; \ x_5^* = 1; \ x_6^* = 0; \ Z^* = \theta \)

b) \( w_1^* = 2; \ w_2^* = 0; \ w_3^* = 5; \ w_4^* = 0; \ w_5^* = 1; \ w_6^* = 0; \ w^*b = \theta \)

What is the value of \( \theta \)?

\[ Z^* = \theta = c'\beta^{-1}b \]

\[ = (c_1, c_3, 0) \begin{bmatrix} 2 \\ 1.5 \\ 1 \end{bmatrix} = 2c_1 + \frac{3}{2}c_3 \]

\[ \theta = w^*b = (2, 0, 5) \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} b_1 + b_3 \\ b_1 + 4b_3 \\ -b_1 + b_2 + 6b_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1.5 \end{bmatrix} \]

\[ \Rightarrow b_1 = \frac{13}{14}, \ b_2 = \frac{15}{14}, \ b_3 = \frac{1}{7} \]

\[ \theta = w^*b = (2, 0, 5) \begin{bmatrix} \frac{13}{14} \\ \frac{15}{14} \\ \frac{1}{7} \end{bmatrix} = \frac{36}{14} = 2 \frac{4}{7} \]

\( c) \ \frac{\partial z}{\partial b_1} = w_1 = 2 – \text{As } b_1 \text{ changes, } z \text{ changes by } 2; \text{ marginal rate of change of } z \text{ with respect to } b_1 \)

\( d) \text{ find } \frac{\partial x_1}{\partial x_6} \)

from the tableau, we have the constraint (1)

\[ x_1 + x_2 + 2x_4^s + x_6^s = 2 \]

or

\[ x_1 = 2 - (x_2 + 2x_4^s + x_6^s) \]

\[ \frac{\partial x_1}{\partial x_6} = -1 \]

e) No, because the shadow price for resource 1 is \( w_1^* = 2 \), so for 1 unit of additional resource, we would expect to increase (a profit) by 2, but the cost is 2.5, so it is not worth it. (Fair price is 2)

f) At optimal solution; \( \frac{\partial z}{\partial b_3} = w_3^* = 5 \), so a “fair price” is 5.
g) Since \( z_2 - c_2 = 0 \) and \( x_2 \) is nonbasic, there is an alternative optimum

Let \( x_2 \) enter, \( x_1 \) leaves,

\[
\begin{array}{c|cccccccccc}
 & Z & x_1 & x_2 & x_3 & x_4^b & x_5^s & x_6^s & \\
\hline
Z & 1 & 0 & 0 & 0 & 2 & 0 & 5 & \\
x_2 & 0 & 1 & 1 & 0 & 2 & 0 & 1 & 2 \\
x_3 & 0 & 0 & 0 & 1 & 1 & 0 & 4 & 1.5 \\
x_5 & 0 & 2 & 0 & 0 & 3 & 1 & 8 & 5 \\
\end{array}
\]

Another optimal solution is:

\( X_1^* = 0; \; X_2^* = 2; \; X_3^* = 1.5; \; X_4^* = 0; \; X_5^* = 5; \; X_6^* = 0; \; Z^* = 0 \)

h) find \( \theta \)

An alternative way to find this from part a & b& is to use

\[
c_B B^{-1} = w \quad \text{(instead of} \quad B^{-1}b = x_B)\]

\[
= (c_1, c_3, 0) \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 4 \\ -1 & 1 & 6 \end{bmatrix} = [2c_1 + c_3, 0, c_1 + 4c_3] = [2, 0, 5]
\]

\[
\begin{align*}
2c_1 + c_3 &= 2 \\
c_1 + 4c_3 &= 5
\end{align*}
\]

\[
\Rightarrow c_1 = \frac{3}{7}, \; c_3 = \frac{8}{7}
\]

\[
\theta = c_B^* b = \begin{bmatrix} \frac{3}{7}, 0, \frac{8}{7} \end{bmatrix} \begin{bmatrix} 2 \\ 1.5 \\ 1 \end{bmatrix} = 2\frac{4}{7}
\]
6.30 (i) \[
\begin{array}{cccccc|c}
\hline
x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\
\hline
0 & 4 & 6 & 18 & 0 & 0 \\
0 & -2 & 0 & -3 & 1 & 0 \\
0 & 0 & -3 & 2 & 0 & 1 \\
\hline
\end{array}
\]

(ii) \[
\begin{array}{cccccc|c}
\hline
x_4 & 0 & 0 & 0 & 0 & 0 \\
\hline
x_5 & -2 & 0 & -3 & 1 & 0 \\
0 & 0 & 2 & 0 & -1 & 0 \\
\hline
\end{array}
\]

(iii) \[
\begin{array}{cccccc|c}
\hline
x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\
\hline
0 & 0 & 8 & 2 & 2 & -16 \\
0 & 1 & 3/2 & -1/2 & 0 & 3/2 \\
0 & 0 & 1 & 2/3 & 0 & -1/3 \\
\hline
\end{array}
\]

6.34 D. \[\text{max } wb \quad \text{s.t. } WA \leq c, \quad w \text{ unrestricted}.\]

Suppose that \(b_r < 0\), and \(y_{ri} \geq 0\) for all \(i\). Consider the vector \((0, 0, \ldots, -1, 0, \ldots, 0) B^{-1}\), where

-1 appears at the \(r\)th position. This vector is the negative \(r\)th row of \(B^{-1}\).

Let \(w\) be any feasible solution of the dual problem, i.e., \(WA \leq c\). Then

\[
(w + \lambda (0, \ldots, 0, -1, 0, \ldots, 0) B^{-1})A = WA + \lambda (0, \ldots, 0, -1, 0, \ldots, 0) B^{-1}A.
\]

By assumption, \((0, \ldots, 0, -1, 0, \ldots, 0) B^{-1}\) is feasible for all \(\lambda > 0\). Therefore, it is a direction of the feasible dual region.

Furthermore, \((0, \ldots, 0, -1, 0, \ldots, 0) B^{-1}b = -b_r > 0\). Hence, we find a direction \(w\) of the feasible dual region such that \(wb > 0\). \(\implies\) The dual problem is unbounded.
Problem 6.46

(P)
Min \(-X_1 - 6X_2\)
Subject to
\[X_1 + X_2 - X_3^S = 2 \quad \Leftrightarrow \quad w_1\]
\[X_1 + 2X_2 + X_4^S = 3 \quad \Leftrightarrow \quad w_2\]
\[X_1, X_2, X_3^S, X_4^S \geq 0\]

(D)
Maximize \(2w_1 + 3w_2\)
Subject to
\[w_1 + w_2 \leq -1 \quad \Leftrightarrow \quad X_1\]
\[w_1 + 2w_2 \leq -6 \quad \Leftrightarrow \quad X_2\]
\[w_1 \geq 0; \ w_2 \leq 0 \quad \} \quad -w_1 \leq 0 \Leftrightarrow X_3 \ ; \ w_2 \leq 0 \Leftrightarrow X_4\]

An initial dual feasible solution (by inspection) is \(w = (0, -3)\)
\[wA - c \leq 0\]
\[(0, -3)\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -1 \\ -6 \end{bmatrix} = [-2 \ 0]\]

The second constraint in (D) is binding, so \(X_2\) is (by complementary slackness) allowed to be positive.

Also, \(w_1 = 0\), so, that constraint is binding, and \(X_3\) is also allowed to be positive

Try to find a feasible solution to (P) using \(Q = \{2, 3\}\)
(by inspection , if \(X_1 = 0\) and \(X_4^S = 0\), \(X_2\) and then \(X_3^S = -0.5\), so its not a feasible solution)

The Phase 1 problem becomes:
Min \(\sum_{j \in Q} 0X_j + 1X^a \Rightarrow \) Min \(X_5^a + X_6^a\)
ST
\[\sum_{j \in Q} a_j X_j + IX^a = b \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} x_5^a \\ x_6^a \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}\]
\[x_2 - x_3^S + x_5^a = 2\]
\[2x_2 + x_6^S = 3\]
\[x_2, x_3^S, x_5^a, x_6^a \geq 0\]

Restricted Primal (Phase 1) problem :

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<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(x_6)</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
The optimal solution to the above is:

\[(X_2, X_3^S, X_5^S, X_6^S) = (1.5, 0, 0.5, 0)\]
with \(X_0 = 0.5\)

Because \(\hat{z}_j - \hat{c}_j < 0\) for \(j \in\) Restricted Problem

The dual of the above restricted primal is:

Min \(2v_1 + 3v_2\)
Subject to \(v_1 + v_2 \leq 0 \iff X_2\)
\(-v_1 \leq 0 \iff X_3^S\)
\(v_1 \leq 1 \iff X_5^S\)
\(v_2 \leq 1 \iff X_6^S\)
\(v_1, v_2\) unrestricted

since \(X_5^S\) and \(X_2\) are basic, first and third dual constrains must be binding

So

\(v_3^* = 0, \ v_5^* = 0\)
\(\iff v_1^* = 1, \ v_2^* = -0.5\)

Then \(v^a_1 = 0.5, \ v^a_2 = 0, \ v^a_3 = -1, \ v^a_4 = 0.5\)

and then \(\theta = \min \left\{ \frac{-2}{0.5} \right\} = 4\)

\(A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}\)
new \(w^I = w + \theta v^* = (0, -3) + 4 [1, -0.5] = [4, -5]\)

The first & second constraints in the dual are binding
so \(Q = \{1, 2\}\)

the new primal problem

Min \(X_5^a + X_6^a\)
Subject to \(X_1 - X_2 + X_5^a = 2\)
\(X_1 + 2X_2 + X_6^a = 3\)
\(X_1, X_2, X_5^a, X_6^a \geq 0\)
The optimal solution to above is:
\((X_1, X_2, X_5^a, X_6^a) = (1, 1, 0, 0)\) with \(x_0 = 0\)

Because \(\hat{z}_j - \hat{c}_j < 0\) for \(j \in \) Restricted Problem

Both \(\hat{z}_j - \hat{c}_j < 0\)

and \(z_j - c_j < 0\)

OPTIMAL SOLUTION IS:

\((X_1, X_2, X_3^s, X_4^s) = (1, 1, 0, 0)\)

\(z = -7\) (or +7 for max)

\((w_1, w_2) = (4, -5)\)

Both primal and dual solutions are feasible and optimal!
Problem 6.48

(P)  
Min \[ x_1 + 2x_2 - x_4 \]  
Subject to \[ x_1 + x_2 + x_3 + x_4 + x_5^S = 6 \] \( \iff \) \( w_1 \)  
\[ 2x_1 - x_2 + 3x_3 - 2x_4 - x_6^S = 5 \] \( \iff \) \( w_2 \)  
x_1, x_2, x_3, x_4, x_5^S, x_6^S \geq 0

(D)  
Maximize \[ 6w_1 + 5w_2 \]  
Subject to \[ w_1 + 2w_2 \leq 1 \] \( \iff \) \( X_1 \)  
\[ w_1 - w_2 \leq 2 \] \( \iff \) \( X_2 \)  
\[ w_1 + 3w_2 \leq 0 \] \( \iff \) \( X_3 \)  
\[ w_1 - 2w_2 \leq -1 \] \( \iff \) \( X_4 \)  
\[ w_1 \leq 0 \] \( \iff \) \( X_5 \)  
\[ -w_2 \leq 0 \] \( \iff \) \( X_6 \)  
w_1; w_2 \text{ unrestricted}

An initial dual feasible solution (by inspection) is \( w = (0, 0.5) \)

\[ w_A - c = (0, 0.5) \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & -1 & 3 & -2 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 2 & -1 & 0 & 0 \end{bmatrix} = 0.5 \]

\[ c_B B^{-1} b = wb = (0, 0.5) \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \frac{5}{2} \]

The 1st, 4th, and 5th dual constraints are binding so \( Q = \{1, 4, 5\} \)

\[ \begin{array}{c|cccccccc} & x_1 & x_2 & x_3 & x_4 & x_5^S & x_6^S & x_7^a & x_8^a \\ \hline Z & 0 & -0.5 & -0.5 & 0 & 0 & -0.5 & 0 & 0 \\ x_0 & 3 & 0 & 4 & -1 & 1 & 0 & 0 & 2.5 \\ x_7^a & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 11 \\ x_8^a & 2 & -1 & 3 & -2 & 0 & -1 & 0 & 6 \end{array} \]

\( \iff w_A - c \)
Let $x_1$ enter and $x_8^a$ leave;

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5^S$</th>
<th>$x_6^S$</th>
<th>$x_7^a$</th>
<th>$x_8^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>0</td>
<td>-0.5</td>
<td>-0.5</td>
<td>0</td>
<td>0</td>
<td>-0.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_0$</td>
<td>0</td>
<td>1.5</td>
<td>-0.5</td>
<td>2</td>
<td>1</td>
<td>0.5</td>
<td>0</td>
<td>-1.5</td>
</tr>
<tr>
<td>$x_7^a$</td>
<td>0</td>
<td>1.5</td>
<td>-0.5</td>
<td>2</td>
<td>1</td>
<td>0.5</td>
<td>1</td>
<td>-0.5</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>-0.5</td>
<td>1.5</td>
<td>-1</td>
<td>0</td>
<td>-0.5</td>
<td>0</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Q for those with $z_j - c_j = 0$

Let $x_5^S$ enter and $x_1$ leave;

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3^S$</th>
<th>$x_4^S$</th>
<th>$x_5^S$</th>
<th>$x_6^S$</th>
<th>$x_7^a$</th>
<th>$x_8^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>0</td>
<td>-0.5</td>
<td>-0.5</td>
<td>0</td>
<td>0</td>
<td>-0.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-0.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_5^S$</td>
<td>0</td>
<td>1.5</td>
<td>-0.5</td>
<td>2</td>
<td>1</td>
<td>0.5</td>
<td>1</td>
<td>-0.5</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>-0.5</td>
<td>1.5</td>
<td>-1</td>
<td>0</td>
<td>-0.5</td>
<td>0</td>
<td>0.5</td>
</tr>
</tbody>
</table>

$x^* = (x_1, x_2, x_3, x_4, x_5^S, x_6^S) = (2.5, 0, 0, 3.5, 0)$

$z^* = 2.5$
Problem 6.18

Min $cx$
ST $Ax = b$
$l \leq x \leq u$

a) (P)
Min $cx$
ST $Ax = b$ ⇔ $w_A$
$x \geq l$ ⇔ $w_l$
$-x \geq -u$ ⇔ $w_u$
$x$ is unrestricted in sign

(D)
Max $w_A b + w_l + w_u$
ST $w_A b + w_l - w_u = c$
$w_A$ unrestricted in sign
$w_l \geq 0$, $w_u \geq 0$

b) the dual always possesses feasible solution.
For example.
let $w_A = 0$, then $w_l - w_u = c$

if $c$ is $\geq 0$, let $w_l = c$, and $w_u = 0$
if $c$ is $< 0$, let $w_u = 0$, and $w_u = |c|$

another way to write this is:
$w_{ij} = \max (0, c_j)$, $w_{ui} = \max (0, -c_j)$

c) If the primal possesses a feasible solution, and from (6) we know the dual possesses a feasible solution, then neither problem can be unbounded and so both problems have finite optimal solutions with equal objectives.