Introduction to Index Notation for Cartesian Vector Representations

We have now spent several weeks dealing with vector quantities using the most direct and explicit notation available. However, such explicit forms can be cumbersome for many calculations. (For example, compute $\vec{a} \times (\vec{b} \times \vec{c})$ and compare with the version below. This handout presents an introduction to manipulating vectors (represented in Cartesian coordinate systems) using index notation. Useful symbols include:

Kronecker Delta: $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

Alternator: $\varepsilon_{ijk} = \begin{cases} 1, & (i,j,k) \text{ cyclic, i.e. } (1,2,3), (2,3,1), (3,1,2) \\ -1, & (i,j,k) \text{ anticyclic, i.e. } (3,2,1), (2,1,3), (1,3,2) \\ 0, & \text{if any two indices are equal} \end{cases}$

In index notation, a vector is represented by its typical component: $\vec{a} = a_i$; $\vec{b} = b_i$.

The scalar (dot, inner) product and cross (vector) product are written as:

$\vec{a} \cdot \vec{b} = a_i b_i$ and $\vec{a} \times \vec{b} = a_i b_j \varepsilon_{ijk}$

where repeated subscripts are automatically summer from 1 to 3 (Einstein Summation Convention); e.g. $a_i b_i = a_i b_i + a_2 b_2 + a_3 b_3$.

Useful properties of the alternator:

$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$

Repeated subscripts are dummies and may be replaced by another letter without affecting the value of the sum: $a_i b_i = a_j b_j$.

Note that if $\delta_{ij}$ appears in an expression summing both $i$ and $j$, then $j$ can be set equal to $i$ and $\delta_{ij}$ removed: $a_i b_j \delta_{ij} = a_i b_i$.

Partial derivative with respect to $x_i$ is abbreviated as: $\partial / \partial x_i = \partial_i$.

Operators involving derivatives (grad, div, curl) become:

$\nabla \cdot \phi = \partial_i \phi$, $\nabla \times \vec{u} = \nabla \cdot \vec{u}$, $\nabla \times \nabla \phi = \partial^2 \phi$.

Example: Show $\nabla \times \nabla \phi = 0$.

$\nabla \times \nabla \phi = \partial_j \partial_i \phi \varepsilon_{ijk} = \partial_j \partial_i \phi \varepsilon_{ijk}$ (change order of differentiation)

Rename dummy indices: $i \rightarrow j$, $j \rightarrow i$

$= -\partial_j \partial_i \phi \varepsilon_{ijk}$ (exchanging adjacent alternator indices changes sign)

$= -\nabla \cdot \nabla \phi = 0 \quad \vec{u} = -\vec{u} \Rightarrow \vec{u} = \vec{0}$

**Exercises** - Derive the following relations using index notation:

1) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

2) $\nabla \cdot (\nabla \times \vec{u}) = 0$

3) $\nabla \times (\nabla \times \vec{u}) = \nabla (\nabla \cdot \vec{u}) - \nabla^2 \vec{u}$ (In the last term, the Laplacian acts on each component of the vector.)