HomeWork 4

Problem 1 (10 points)

Using the same matrix as in Homework 3, but with N=50

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\ldots & \ldots & \ldots & \ldots & -1 & 1
\end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}
\]

Compute the solution using the SOR iterative method as a function of \( \omega \). Let \( \omega \) range from 1.2 to 1.95. Plot the number of iterations needed. Convergence is defined when the maximum mean squared residual is less than \( 10^{-8} \). Discuss the effect of \( \omega \).

Problem 2 (20 points)

We want to integrate

\[
\int_0^1 \frac{x}{\sqrt{1-x^2}} dx
\]

Now this integrand has a singularity at x=1 because the denominator \( \sqrt{1-x^2} = \sqrt{(1-x)(1+x)} \).

a. The only way to handle this when using an integration method that involves evaluating the integrand at the end points, is to integrate from \( 0 \leq x \leq 1-a \) numerically and from \( 1-a \leq x \leq 1 \) analytically. In the range \( 1-a \leq x \leq 1 \) let \( y = 1-x \). You will end up with an integrand involving \( \sqrt{2-y} \) in the denominator. Express this term as a Taylor series and integrate analytically. The question is what should \( a \) be and how many terms to use in the Taylor series.

Use composite Simpson’s rule with \( N \) panels to integrate from \( 0 \leq x \leq 1-a \) and your Taylor series result for \( 1-a \leq x \leq 1 \) to get the answer which is 1. See what value of \( N \) is required to get good answers as a function of \( a \) and the number of terms in the Taylor series.

b. Now Gaussian Quadrature does not evaluate the integrand at the end points, so you might consider one of the different formulae. Remember that for integrating between fixed limits that you must convert them to \( -1 \leq x \leq 1 \). After converting to these limits, your integrand will not match any of the formulae listed in the notes. However, it appears that the Chebyshev formula will handle the singularity at one end, but we don’t have a singularity at \( x = 0 \). Let’s try it anyway because it is the simplest to use. The weights are all equal to \( \pi/n \). The point to evaluate the integrand at are \( x = \cos((2i-1)\pi/2n), i = 1, \ldots, n \). Using Gauss-Chebyshev quadrature requires that the integrand be of the form \( f(x)/\sqrt{1-x^2} \) and the result is \( \int f = \sum w_i f(x_i) \) so you will have to put your integrand in this form. How many points must you use to get a good answer?