The test is 2.0 hour, open book/notes exam. You can do it anytime during the weekend but it must be submitted by 9AM Monday, May 4th. You can either submit a paper copy to the ME front office, slip it under my office door, or email a pdf to me.

Please put your name on the upper right corner of the exam with the time that you started and ended.

**Question 1** Consider the infinite series

\[ \Phi(x) = \sum_{k=1}^{\infty} \frac{1}{k(k+x)} \]  

1. Using the same approach as in Homework 1, determine how many terms must be computed to obtain an accuracy of \(10^{-8}\).

2. It can be shown that \(\Phi(1) = 1\). Consider the infinite series

\[ \Phi(x) - \Phi(1) \]

and determine how many terms must be needed. Does machine epsilon affect the result?

**Question 2** If we fit \(f(x)\) with a \(n^{th}\) order polynomial we have

\[ f(x) = p_n(x) + E_n(x), \quad p_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \]  

where

\[ E_n(x) = \frac{\prod_{j=0}^{n} (x - x_j)f^{(n+1)}(\xi)}{(n+1)!}, \quad \xi \text{ is in the range } [x_0, \ldots, x_n, x] \]

1. given values of \(f(x)\) at two points, \(f(x_1)\) and \(f(x_2)\), determine the 1st order polynomial, \(p_1(x)\) and the error \(E_1(x)\).

2. write the expression for \(\int_a^b f(x)dx\) and for the integral of \(E_1\) where the points \(x_1\) and \(x_2\) are located between \(a\) and \(b\) but do not coincide with the points \(a\) and \(b\).

3. write the expression for \(\int_a^b f(x)dx\) and for the integral of \(E_1\) when \(x_1 = a, x_2 = b\).

4. We know that there is one and only one \(n^{th}\) order polynomial that fits \(n+1\) data points. The polynomial can be expressed as a Lagrange polynomial

\[ f(x) = \sum_{j} l_j(x)f(x_j) \]  

Comparing Eqs.3a and 4 write the expressions for \(l_1(x)\) and \(l_2(x)\)
5. An improved trapezoidal integration is given by

\[ \int_{x_1}^{x_2} f(x) \, dx = \frac{h}{2}(f(x_1) + f(x_2)) + \frac{h^2}{12}(f'(x_1) - f'(x_2)) \]  

(5)

where \( h = x_2 - x_1 \). Interpolations that involve the derivatives are called "Hermite Interpolations" and their integral is expressed as

\[ \int_{x_1}^{x_2} f(x) \, dx = \sum H_j f(x_j) + \sum \overline{H}_j f'(x_j) \]

(6)

where

\[ \overline{H}_j = \int_{x_1}^{x_2} (x - x_j) l_j^2(x) \, dx \]  

(7)

using the expressions for \( l_1(x) \) and \( l_2(x) \) that you found in 4, show that \( H_1 = \frac{h^2}{12}, H_2 = -\frac{h^2}{12} \).

**Question 3**

Let us fit data by \( f(x) = \sum_{j=1}^{M} a_j \phi_j(x) \) and evaluate the coefficients \( a_j \) using the least squares method. These coefficients are found by solving the set of simultaneous linear equations \( Aa = b \) where the matrix \( A \) has elements \( \alpha_{jk} \). Let \( x \) be 21 equally spaced data points between 0 ≤ \( x \) ≤ 1. We have

\[ \sum_{k=1}^{M} \alpha_{jk} a_j = b_j, \quad j, k = 1, \cdots, M \]  

(8a)

where

\[ \alpha_{jk} = \sum_{i=1}^{N} \phi_j(x_i) \phi_k(x_i) \]

\[ b_j = \sum_{i=1}^{N} f(x_i) \phi_j(x_i) \]  

(8b)

1. Fit all the data points with a \( M^{th} \) order polynomial. Write a Matlab program to evaluate the matrix \( A \) for 0 ≤ \( M \) ≤ 10. Evaluate the condition number. What conclusion do you draw?

2. Consider the case where \( x \) is uniformly distributed between 0 and 2\( \pi \) and \( \phi_j(x) = \cos(jx) \) and for which we have the orthogonality relationship \( \sum \cos(jx_i)\cos(kx_i) = 0 \) if \( j \neq k \). Show that the \( A \) matrix is diagonal. What do you conclude?

3. Let us fit using the Legendre orthogonal polynomials, \( \phi_1(x) = 1, \phi_2(x) = \sqrt{3}s, \phi_3(x) = \sqrt{(5)/2}(3s^3 - 1), \phi_4(x) = \sqrt{7}/2(5s^3 - 3s) \) where \( s = 2x - 1 \). These polynomials satisfy \( \int \phi_i(x)\phi_j(x) \, dx \) over 0 ≤ \( x \) ≤ 1 but are
not orthogonal for discrete data points. Since a sum over discrete \( x_i \) is equal to the integral in the limit, we hope that the \( A \) matrix might be better behaved for large number of sample points. For the 21 data points uniformly distributed between 0 and 1, compute the \( A \) matrix and its condition number. Is it better behaved?

**Question 4** Minimizing or maximizing a function is often done using Lagrange multipliers. Consider finding the point on the curve \( y^2 = 4x \) which is closest to the point \( x = 1, y = 0 \). This means minimizing

\[
 s^2 = (x - 1)^2 + y^2 \quad (9a)
\]

subject to

\[
 \phi = y^2 - 4x = 0 \quad (9b)
\]

1. Method 1: from \( \phi = 0 \) solve for \( y \) and substitute into Eq. 9a giving \( s^2 \) as a function of \( x \) only and by differentiating with respect to \( x \) determine the value of \( x \) that minimizes \( s^2 \) and then from Eq. 9b the value of \( y \)

2. Method 2: from \( \phi = 0 \) solve for \( x \) and substitute into Eq. 9a giving a function of \( y \) only and by differentiating with respect to \( y \) determine the value of \( y \) that minimizes \( s^2 \) and then from Eq. 9b the value of \( x \)

3. Method 3: Using Lagrange multipliers allows us to differentiate with respect to \( x \) and \( y \) in minimizing the function

\[
 F = s^2 + \lambda \phi \quad (10)
\]

Differentiate \( F \) with respect to \( x, y, \) and \( \lambda \). Setting these derivatives equal to 0, gives us three equations for \( x, y, \) and \( \lambda \). Solve for the value of \( x, y, \) and \( \lambda \).

Comment on the three methods. Which is preferable?

Solving the above problem was easy because Eq. 9b allowed us to express \( x \) in terms of \( y \) or \( y \) in terms of \( x \). Sometimes the constraints can be more complex. Consider determining the dimensions of a rectangular box with dimensions \( L, W, H \) of a specified volume \( V \). Find the dimensions if the box has an open top and we want the surface area to be a minimum by using Lagrange multipliers. Does the open top have a strong effect on the answer?