Root Finding

Before trying to find \( x \), you should

1) Understand the meaning of \( f(x) \)
2) Know what any parameters represent
3) Visualize \( f(x) \)
4) Define \( x_1 \) & \( x_2 \), values of \( x \) that straddle the root ! !

4) is a requirement - you cannot proceed unless you have any idea where the root is

Methods

A) Bisection

replace either \( x_1 \)
or \( x_2 \) with \( x_3 \), where

\[ x_3 = \frac{1}{2} (x_1 + x_2) \]

the root lies in the region

\[ r_1 = |x_2 - x_1| \]
\[ r_2 = |x_2 - x_3| \] & \( r_3 = \frac{r_1}{2} \)

like wise

\[ r_3 = \frac{r_2}{2} \] , \( r_2 = \frac{r_1}{2^{n-1}} \)

the method

a) we can determine \( n \) before finding \( x \)
b) terminates when \( r < \) multiple of mach c

c) can handle discontinuities
B. False Position - always straddle the root

Here we assume that $f(x)$ is linear between $x_1$ and $x_2$

$$x_3 = \frac{y_2 x_1 - y_1 x_2}{y_2 - y_1}$$

If $x_3 > x_1$, then

$$x_4 = \frac{y_3 x_1 - y_1 x_3}{y_3 - y_1}$$

Note that the form of the equation has changed. The method is not stationary.

C. Secant Method - do not always straddle $a$

Here we use the stationary method

$$x_{i+1} = \frac{y_i x_{i-1} - y_{i-1} x_i}{y_i - y_{i-1}}$$

This method may fail to converge by sending you off to $\infty$.

It is best to use

$$x_{i+1} = x_i - \frac{x_i - x_{i-1}}{y_i - y_{i-1}} y_i$$

to avoid numerical difficulties as $y_i \rightarrow 0$.
Order of the Method

Let \( E_i = \alpha - x_i \).

If
\[
\lim_{i \to \infty} \frac{|E_{i+1}|}{|E_i|^p} = C \neq 0
\]
the method is said to be of order \( p \).

And \( C \) is the asymptotic error constant.

If \( p = 1 \), \( C < 1 \) \}
\[ p > 1 \), \( C \) is not restricted \} convergence

Computational Cost

Let us consider 2 methods \( a \) and \( b \):

\[
|E_{b+1}^a| = C^a |E_i^a| \quad |E_{b+1}^b| = C^b |E_i^b|^p
\]

If \( S_i = -\ln |E_i^a| \) \& \( T_i = -\ln |E_i^b| \)

then \( S_{b+1}^a = -\ln C + \frac{P_a}{P_a S_o} \quad T_{b+1}^b = -\ln C + \frac{P_b}{P_b T_o} \)

These two difference equations have the solutions:

\[
S_i = S_i^0 (P_a)^{i-1} - \left( \frac{P_a^{i-1}}{P_a - 1} \right) \ln C_a
\]

If \( S_i = T_i \) \& \( S_i^J = T_J \), i.e. errors are the same initially \& at convergence

then
\[
S_i \left( \frac{P_a^J - P_b^J}{P_a^J - 1} \right) + \left( \frac{P_b^{J-2} - 1}{P_b^{J-2} - 1} \right) \ln C_b = 0
\]

after \( I \) iterations with method \( a \) \&

\( J \) iterations with method \( b \)
Let $\Theta$ be the cost of a computation, then the costs are $\zeta \Theta_a$ and $\zeta \Theta_b$.

Many times the second term is small and we have

\[ \zeta_a = \zeta_b \quad \text{and} \quad \zeta \ln \zeta_a = \zeta \ln \zeta_b \]

and $\zeta = \frac{\ln \zeta_b}{\zeta}$.

This suggests that the total cost is $\propto \frac{\Theta}{\ln p}$.

So we define the efficiency as proportional to $1/$cost. $\propto \frac{\ln p}{\Theta}$

So our efficiency $\propto p \frac{1}{\Theta}$ - valid only near $\Theta$.

How do we find $p$?
Inverse Interpolation

If \( y = f(x) \), let \( x = g(y) = \sum_{j=1}^{n} \ell_j(y) g(y_j) \)

with \( a = g(0) \), then our new approximation will be

\[
x_{i+1} = \sum_{j=1}^{n} \ell_j(a) g(y_j)
\]

(1)

where \( \ell_j(a) = \frac{(-1)^{j-1} y_1 y_2 \ldots y_n}{(y_j-y_1)(y_j-y_2)\ldots(y_j-y_n)} \)

the error in the approximation is

\[
x_n x_{i+1} = \sum_{j=1}^{n} \ell_j(y) g(y_j) (-1)^{n-1} y_1 y_2 \ldots y_n
\]

where \( y_n \) is in the interval \( y_1, y_2, \ldots, y_n \).

Equation (1) is of the form

\[
x_{i+1} = F_c(x_0, x_{i-1}, \ldots, x_{i-n+1})
\]

If \( F_c = F \), a constant function, method is stationary, e.g. secant method.

If \( F_c \) changes with \( i \), method is non-stationary, e.g. bisection, false position.

In general \( F_c \) may involve \( f(x) \) or derivatives.
False Position - Order of

Suppose that our \( f(x) \) gives rise to a stationary process

\[ X_{n+1} = F(X_n, X_1) \]

then \( E_n = x - X_{n+1} = \frac{g''(y)}{2} y_c y_1 \)

Now \( x = g(y) \), \( dx = dg \), \( dy = \frac{dx}{dy} dy = \frac{1}{f'} dy \)

Then \( g'' = -\frac{f''}{(f')^3} \)

Now \( y_i = f(x_i) = f(x_i) - f(x) = (x_i - x) f'(x_i) \)
\( y_c = f(x_c) = f(x_c) - f(x) = (x_c - x) f'(x_c) \)

Giving \( \frac{E_n}{E_i} = \frac{-\frac{f''(x)}{2} f'(x_i) f'(x_c)}{(f'(x))^3} E_c E_i \)

And \( \lim_{i \to \infty} \frac{E_{n+1}}{E_i} = -\frac{f''(x)}{2 (f'(x))^3} \)

Thus \( p = 1 \) linear convergence
\( \Gamma = 1 \) assuming \( \Theta = 1 \)
Secant Method – Order of

Here we have the stationary method
with

\[ E_{n+1} = -\frac{f''(x_0) f'(x_n) f'(x_{n-1}) E_n}{2 (f'(x))^3} \]

If \( f'(x) \) and \( f''(x) \) are bounded

\[ \lim_{n \to \infty} \frac{|E_{n+1}|}{|E_n|^p} = \frac{|f''(x)|}{2 |f'(x)|} < C^q \]

Where \( p = 1 + \frac{\sqrt{5}}{2}, \quad q = \frac{\sqrt{5} - 1}{2} \)

\[ \approx 1.62 \]

and the method is more efficient than bisection or false position

but it may diverge.

Therefore it must be combined with always straddling the root.
One Point Methods - Newton-Raphson

\[ X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)} \]

Let \( X_{n+1} = F(x_n) \) at convergence \( x = F(x) \)

\[ F(X_n) = F(x) + (X_n - x) F'(x) + \frac{(X_n - x)^2 F''(x)}{2} + \ldots \]

\[ X_{n+1} - x = (X_n - x) F'(x) + \frac{(X_n - x)^2 F''(x)}{2} + \ldots \] \((A)\)

If \( F^{(n)}(x) = 0 \) \( 1 \leq n \leq p \) the method is of order \( p \)

That is \( X_{n+1} - x = (X_n - x)^p \frac{F^{(p)}(x)}{p!} \)

**Newton's Method**

\[ F(x_n) = X_n - \frac{f(x_n)}{f'(x_n)} \]

\[ f'(x_n) = 1 - \frac{f'(x_n) + f(x_n)f''(x_n)}{f''(x_n)} = 0 \]

\[ f''(x_n) \]

\[ - \frac{f''(x_n)}{f'(x_n)} = f'(x_n) f'(x_n) \]

\(*\)

\[ e_{n+1} = X_{n+1} - x = \frac{e_n^2 F''(x_n)}{2} \]

\[ \frac{e_n^2 F''(x_n)}{2} \leq p = 2 \]

**Modified Newton's Method**

Use a 2 point Hermite interpolation

\[ f(x_n) = \frac{1}{(x_n - x_{n-1})^2} \left[ \left(1 - \frac{2(x - x_n)}{X_{n-1} - X_n} \right) (x - x_{n-1})^2 f(x_n) \right. \]

\[ + \left(1 - \frac{2(x - x_{n-1})}{X_{n-1} - X_n} \right) (x - x_n)^2 f(x_{n-1}) \]

\[ + (x - x_n)(x - x_{n-1})^2 \frac{f'(x_n)}{f'(x_n)} + (x - x_{n-1})(x - x_n)^2 \frac{f'(x_{n-1})}{f'(x_{n-1})} \]

leads to

\[ X_{n+1} = X_n - \frac{f(x_n) - \frac{1}{2} \frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3}}{f'(x_n)} \]

\[ + \frac{f'(x_n)}{f'(x_n)} \]

where \( \frac{f'(x_n)}{n_n^2} \)

\[ \frac{f(x_n) - f(x_{n-1})}{h_n} + 2 \left( \frac{f'(x_n) + f'(x_{n-1})}{h_n} \right) \]

where \( h_n = X_n - x_{n-1} \) \( \leq p = 1 + \sqrt{3} > 2 \)
Multiple Roots

Clearly \( f(x) = 0 \), \( a_1 < x < a_2 \). If \( a_1 = a_2 \), \( f'(x) = 0 \), \( f''(x) \neq 0 \).

From (A)

\[
E_{n+1} = E_n \left( 1 - \frac{f'(x)}{f''(x)} \right) \cdots
\]

Newton's Method

\[
F(x) = x - \frac{f(x)}{f'(x)}
\]

\[
F'(x) = 1 - \frac{f'(x)}{f''(x)} = \frac{f(x) f''(x) - f'(x)^2}{f''(x)} = g
\]

Using l'Hospital's rule

\[
F'(x) = \frac{f'(x) f''(x) + f(x) f'''(x)}{2 f''(x)^2} = \frac{1}{2} \left( 1 + \frac{f'(x) f'''(x)}{2 f''(x)^2} \right)
\]

Therefore

\[
E_{n+1} = \frac{1}{2} E_n \quad \frac{1}{2} \Rightarrow \quad p = 1
\]

Let \( F(x) = x - \frac{f(x)}{f'(x)} \), where \( r \) is the multiplicity of roots.

\[
F'(x) = 1 - \frac{f'(x)}{f''(x)} + \frac{f(x) f'''(x)}{2 f''(x)^2} \quad \text{and using l'Hospital's rule}
\]

\[
= 1 - \frac{f'(x)}{f''(x)} + \frac{f(x) f'''(x)}{2 f''(x)^2} + \frac{f(x) f'''(x)}{2 f''(x)^2} + \frac{f(x) f'''(x)}{2 f''(x)^2}
\]

\[
F'(x) = 1 - \frac{f'(x)}{f''(x)}
\]

If \( r = 2 \), \( F'(x) = 0 \) \& Newton's method has \( p = 2 \).

*Using l'Hospital's rule shows the last term to be 0.*
Systems of Equations

\[ f_i(x^k) = 0 \quad 1 \leq i \leq k \quad x^k \text{ is a vector} \]

\[ f_i(x^k) = f_i(d^k) + \frac{\partial f_i}{\partial x^k}(x^k) \]

\[ \frac{\partial f_i}{\partial x^k} = \frac{\partial f_i}{\partial x}(x^k) - f_i(x^k) \]

Let \( A = \left[ \frac{\partial f_i}{\partial x^k} \right] \), then

\[ d^k = x^k - A^{-1} f_i(x^k) \]

As for \( k=1 \), the method is of order 2 if \( n=1 \)

But good estimates of \( d^k \) may not be available because it is hard to "stir" the roots.

Convergence

- Monotone convergence
  \[ |f'(x)| \leq 1 \]

- Oscillating convergence

- Divergence
  \[ |f'(x)| > 1 \]
Example: $\cos(x) - x = 0$

$$x_{n+1} = \cos(x_n)$$

$$F'(x) = -\sin(x) \quad |F'(x)| \leq 1 \quad \text{Convergence}$$

Consider:

$$x_{n+1} = (1-k)x_n + kF(x_n)$$

$$\alpha = (1-k)\alpha + k\alpha = \alpha$$

$$f(x) = x^3 - 3x + 1 = 0$$

$$x_{n+1} = \frac{1}{3} (x_n^3 + 1)$$

$$F'(x) = \frac{3x^2}{3} = x^2 \quad F' < 1 \quad x_n < 1 \rightarrow a_2$$

$$a_1, a_2 \text{ diverge}$$

$$x_{n+1} = \frac{3}{2} x_n - \frac{1}{6} (x_n^3 + 1) \quad \text{let } k = -\frac{1}{2}$$

$$F'(x) = \frac{3}{2} - \frac{3}{2} x_n^2 = -\frac{3}{2} x_n^2 \rightarrow a_3, a_4 \text{ diverges}$$