Wavelets

Consider a signal discretely sampled

\[
\begin{align*}
    \text{that we want to represent. Let the sample points cover the range } & \ x_1 < x_2 < L + x_i \\
\end{align*}
\]

then \( L \) will be called the support of our approximation. Different levels of support are often encountered

Fourier series  \( L = \) all points

DFT  \( \sim \)

Splines  \( \sim \)

When \( L = \) all points, we have no way of representing a local region.

Haar Transform

Consider representing a signal by a vector \( f = [f_1, f_2, \ldots, f_N] \)

Let us form 2 vectors by defining

\[
\begin{align*}
    A' &= [\frac{f_1 + f_2}{\sqrt{2}}, \frac{f_2 + f_3}{\sqrt{2}}, \ldots, \frac{f_{N-1} + f_N}{\sqrt{2}}] = [d_1', d_2' \ldots] \\
    D' &= [\frac{f_2 - f_1}{\sqrt{2}}, \frac{f_3 - f_2}{\sqrt{2}}, \ldots, \frac{f_N - f_{N-1}}{\sqrt{2}}] = [d_2', d_3' \ldots]
\end{align*}
\]
or \( \alpha_0' = \frac{f_{26} - f_{20}}{\sqrt{2}}, \quad \alpha_i' = \frac{f_{26-i} - f_{20}}{\sqrt{2}} \)

Then \( f_1 = \alpha_i' + \alpha_i', \quad f_2 = \alpha_i' - \alpha_i' \)

(\( \alpha_i' \) are necessary so that the energy \( \sum f_i^2 = \sum \alpha_i'^2 + \sum \alpha_i'^2 \))

We can represent the Haar transform as

\[ f \xrightarrow{H} (A', D') \]

where \( A' \) is called the "trend" and represents the average or smooth part of the signal.

\( D' \) is called the "detail" or "fluctuations" and represents the local fluctuations.

**Multi level Analysis**

Suppose that we transform the average signal \( A' \) so that

\[ A' \xrightarrow{H} (A^2, D^2) \]

then Eq (1) can be written as

\[ f \xrightarrow{H'} (A'1D') \xrightarrow{H'} (A^21D^2) \]

or \[ f \xrightarrow{H^2} (A^21D^21D') \]
If \( N = 2^m \), we can continue transforming \( m \) times and \( A^m \) will just be a single number representing the average \( \frac{\sum f}{N} \).

**Denoising - Compression**

Now as \( D \) represents fluctuations, if we set \( D' = 0 \) & reconstruct \( f \) we will get a smoother signal which represents the elimination of the local fluctuations.

If the energy of \( D^n \) is small, then we will not have affected our interpretation of the signal.

Since \( D^n = 0 \), we need not transmit it & we have "compressed" the signal into a smaller vector.

Of course we will lose detail so it is important to be careful in selecting the value of \( n \) in eliminating \( D^n \).
Wavelet Examples

Consider the signal shown in Figure 1a that has 2048 sample points. A Haar wavelet analysis was done with 11 levels. Figure 2b depicts the energy contained in the Haar detail vectors. Note that the first several levels had very small energies and could be ignored without seriously affecting the signal. Figure 2 shows the effect of ignoring detail vectors each with less than 1% of the total energy. Notice that when the original signal varies rapidly, the average signal does a good job, but when the signal varies slowly, between 400 and 600, the reconstructed result is not as good.

In this case the 1st 4 detail vectors were ignored. These vectors are 1024, 512, 256, and 128 long, so transmitting the information can be done by sending $2048 - (1024 + 512 + 256 + 128) = 128$ pieces of information, a tremendous saving.

The Haar wavelet is very good at detecting local fluctuations because the components of the detail transform are $d_i = (f_{2i-1} - f_{2i})/\sqrt{2}$, but its average is not good at following trends because of its very local support. Other wavelets are available that do a better job. Three common ones are the 'daub', 'coi' and 'bior' wavelets. The daub and coi wavelets have more support, i.e., sample a wider range of values, and thus follow the trends better but do a poorer job in capturing local fluctuations.

Figure 3 shows a plot of the signal and the 1st, 2nd, and 3rd fluctuations. Figure 4 shows how the approximations agree with the original signal.

Even though the fluctuations shown in Fig. 3 appear to be substantial, their energy content is small, Figure 1b, and ignoring them does not substantially affect our understanding of the signal. On the other hand the last several detail vectors that are more affected by gross fluctuations, vectors $D^6$ to $D_{11}$, have a
dropping 4 details each with less than 0.01 percent of energy

Figure 2: Compressed signal

Figure 3: Signal and the 1st 3 fluctuations

large energy and must be retained. But these vectors are quite short with $D_{11}$ being only one component long.

Figure 5 shows the energy content of the bior wavelet analysis for comparison.
Images

Wavelets are particularly good at denoising and compressing images. Figure 6 shows the effect of compressing an image using the bior3.7 wavelet. The numbers above the figures show the number of vector components that are set to zero and the energy of the compressed signal.
Even the approximations give adequate images, Figure 7. The original image was 256x256 while the level 2 image is 75x75.

Figure 7: Approximate Images using the bior wavelet

and this is due to the lack of detail as shown in Figure 8
Figure 8: Details of Level 1 bior transform