1. This problem deals with combination resonances. Consider the Duffing's equation under excitations of two frequencies, i.e.,

\[
\ddot{u} + \omega_0^2 u + \epsilon \left(2\mu \dot{u} + \alpha u^3\right) = K_1 \cos \Omega_1 t + K_2 \cos \Omega_2 t
\]  

(1)

where

\[
\Omega_1 + \Omega_2 = 2\omega_0 + \epsilon \sigma
\]  

(2)

(a) Use the method of multiple scale to obtain the following secular terms

\[
2i\omega_0 \left(A' + \mu A\right) + \alpha \left(3A\ddot{A} + 6A^2 \dot{A} + 6\dot{A}^2\right) A + 6\alpha \Lambda_1 \Lambda_2 \ddot{A} e^{i\sigma T_1} = 0
\]  

(3)

where

\[
\Lambda_n = \frac{K_n}{2 \left(\omega_0^2 - \Omega_n^2\right)}, \quad n = 1, 2
\]  

(4)

(b) Let

\[
A(T_1) = \frac{1}{2} a(T_1)e^{i(\sigma T_1 - \gamma(T_1))/2}
\]  

(5)

Show that \(a\) and \(\gamma\) are governed by

\[
\frac{da}{dT_1} = -\mu a - \alpha \Gamma_1 a \sin \gamma
\]  

(6)

\[
a \frac{d\gamma}{dT_1} = (\sigma - 2\alpha \Gamma_2) a - \frac{3\alpha a^3}{4\omega_0} - 2\alpha \Gamma_1 a \cos \gamma
\]  

(7)

where

\[
\Gamma_1 = \frac{3K_1 K_2}{4\omega_0 \left(\omega_0^2 - \Omega_1^2\right) \left(\omega_0^2 - \Omega_2^2\right)}
\]  

(8)

\[
\Gamma_2 = \frac{3}{4\omega_0} \left[\frac{K_1^2}{\left(\omega_0^2 - \Omega_1^2\right)^2} + \frac{K_2^2}{\left(\omega_0^2 - \Omega_2^2\right)^2}\right]
\]  

(9)

(c) Show that the steady-state amplitudes are given by

\[
a = 0
\]  

(10)

or

\[
a^2 = \frac{8\omega_0}{3} \left[\frac{\sigma}{2\alpha} - \Gamma_2 \pm \sqrt{\Gamma_1^2 - \frac{\mu^2}{\alpha^2}}\right]
\]  

(11)

2. Consider an undamped pendulum oscillating with small amplitudes governed by

\[
\frac{d^2 x}{dt^2} + \omega_n^2 x = 0, \quad \omega_n = \sqrt{\frac{g}{l}}
\]  

(12)

(a) Determine the fundamental matrix \(\Phi(t)\).
(b) Calculate $\det \Phi(t)$.

(c) Verify the Liouville theorem.

3. Consider a parametrically excited system whose continuous response is governed by

$$\frac{d^2x}{dt^2} + \omega_n^2 x = ax(t) \sum_{k=1}^{\infty} \delta(t - k)$$

where $\delta(t - k)$ is the delta function defined as

$$\delta(t - k) = \begin{cases} 0, & t \neq k \\ \infty, & t = k \end{cases}$$

with

$$\int_{t_1}^{t_2} x(t)\delta(t - k)dt = x(k), \quad \text{if} \quad t_1 < t < t_2$$

This system has period $T = 1$. Determine $\Phi(1^+)$ of this system.