1. Consider a taut elastic string whose tension $T(t)$ is varying with time. The equation of motion of the string is known as

$$T(t) \frac{\partial^2 w}{\partial x^2} = \rho \frac{\partial^2 w}{\partial t^2}$$  \hspace{1cm} (1)

where $w(x,t)$ is the deflection of the string, and $\rho$ is the density per unit length of the string.

(a) If the string is pinned at both ends, i.e., $w(0,t) = w(l,t) = 0$, where $l$ is the length of the string, the deflection $w(x,t)$ can be found through

$$w(x,t) = \sum_{n=0}^{\infty} \left( \sin \frac{n\pi x}{l} \right) q_n(t)$$  \hspace{1cm} (2)

If the tension $T(t)$ is periodic given by

$$T(t) = S(1 + a \cos 2\Omega t)$$  \hspace{1cm} (3)

show that $q_n(t)$ will satisfy the Mathieu equation.

(b) Find the frequency ranges of $\Omega$ in which primary resonances of the string occur.

2. This problem is to generate the Ince-Strutt chart of the damped Mathieu's equation through a numerical method. Consider the damped Mathieu's equation

$$\ddot{x} + 2\mu \dot{x} + (\delta + 2\epsilon \cos 2t) x = 0$$  \hspace{1cm} (4)

Numerically integrate (4) over a period to obtain $\Phi(T)$, where $T = \pi$. Then find the eigenvalues of $\Phi(T)$ to determine the stability of the system. Plot the stability regions for $-1 < \delta < 5$ and $0 < \epsilon < 3$. Plot the Ince-Strutt charts for the following three cases: (a) $\mu = 0$, (b) $\mu = 0.1$, and (c) $\mu = 0.2$.

3. Consider a parametrically excited system whose response is governed by

$$\ddot{x} + \left[ \delta + \epsilon f(t) \right] x = 0$$  \hspace{1cm} (5)

where

$$f(t) = \begin{cases} 
1, & 0 < t < \pi \\
-1, & \pi < t < 2\pi 
\end{cases}$$  \hspace{1cm} (6)

In addition, $f(t + 2\pi) = f(t)$ and $\delta < \epsilon$. Also, $x$ and $\dot{x}$ are continuous at $t = \pi$.

(a) Determine $\Phi(2\pi)$, which is the fundamental matrix evaluated at $T = 2\pi$.

(b) For this parametrically excited system, will there be periodic solutions of period $T$ or $2T$ at the stability boundaries on the $\delta - \epsilon$ parameter plane as in the Mathieu's equation? Justify your answer.

(c) Find the exact stability boundaries on the $\delta - \epsilon$ parameter plane using $\Phi(2\pi)$.

(d) Perform a Linsted-Poncaré analysis to obtain the stability boundaries of the primary resonance to the first order.