Mathematica for Fourier Series and Transforms

Fourier Series

Periodic odd step function

Use built-in function "UnitStep" to define. "Mod" allows one to make the function periodic, with the "-Pi" shifting the fundamental region of the Mod to -Pi to Pi (rather than 0 to 2Pi). The period is taken to be 2 Pi, symmetric around the origin, so the function is even.

"Exclusions->None" makes the plot include the steps.

\[ff0[x_] = -1/2 + UnitStep[Mod[x, 2 Pi, -Pi]];\]

\[ff0plot = Plot[ff0[x], \{x, -3 Pi, 3 Pi\}, Ticks \rightarrow \{-3 Pi, -2 Pi, -Pi, 0, Pi, 2 Pi, 3 Pi\}, \text{PlotStyle} \rightarrow \{\text{Thick, Red}\}, \text{Exclusions} \rightarrow \text{None}]\]

Truncated Fourier Series---since odd can use "FourierSinSeries"

\[\text{In[170]} = \text{ft6ff0} = \text{FourierSinSeries}[ff0[x], x, 6]\]

\[\text{Out[170]} = \frac{2 \text{Sin}[x]}{\pi} + \frac{2 \text{Sin}[3 x]}{3 \pi} + \frac{2 \text{Sin}[5 x]}{5 \pi}\]

Faster for evaluation to define Fourier Series directly (given that we’ve worked it out by hand in class notes):

\[\text{ftff0}[x\_, nmax\_] := \text{Sum}[\text{Sin}[\{2 \text{n} + 1\} x] / (2 \text{n} + 1), \{\text{n}, 0, nmax\}] 2 / \pi\]
Comparing step to truncated Fourier series:

\[
\text{ftff0}[x, 2] = 2 \left( \frac{\sin(x)}{\pi} + \frac{1}{3} \sin(3x) + \frac{1}{5} \sin(5x) \right)
\]

Make animation showing convergence:

\[
\text{Manipulate}[\text{stepplot}[n], \{n, 0, 20, 1\}]
\]
Plot[\{ff0[x], ftff0[x, 2], ftff0[x, 20], ftff0[x, 100]\},
\{x, -Pi, Pi\}, Ticks \rightarrow \{-Pi, -Pi/2, 0, Pi/2, Pi\}, PlotStyle \rightarrow 
\{(Thick, Red), (Thick, Blue), (Thick, Green), (Thick, Black)\}, Exclusions \rightarrow None]

Zoom in to see Gibbs phenomenon: lack of convergence near step.
Here use 21, 101 and 501 term series.

Plot[\{ff0[x], ftff0[x, 20], ftff0[x, 100], ftff0[x, 500]\},
\{x, -0.1, +0.1\}, PlotRange \rightarrow \{-0.1, 0.1\}, (0.4, .6\}, PlotStyle \rightarrow 
\{(Thick, Red), (Thick, Green), (Thick, Black), (Thick, Orange)\}, Exclusions \rightarrow None]

Using complex fourier series

\textbf{In[174]} \textbf{=} \texttt{ft6ff0complex = FourierSeries[ff0[x], x, 6]}

\textbf{Out[174]} \quad \frac{i e^{-ix}}{\pi} - \frac{i e^{ix}}{\pi} + \frac{i e^{-3ix}}{3 \pi} - \frac{i e^{3ix}}{3 \pi} + \frac{i e^{-5ix}}{5 \pi} - \frac{i e^{5ix}}{5 \pi}

Simplify doesn't help, but FullSimplify brings result back into Sin form found above

\textbf{In[178]} \textbf{=} \texttt{FullSimplify[ft6ff0complex]}

\textbf{Out[178]} \quad \frac{2 (15 \sin[x] + 5 \sin[3 x] + 3 \sin[5 x])}{15 \pi}
Even "barrier" function

Use built-in function "UnitStep" to define. "Mod" allows one to make the function periodic, with the "-Pi" shifting the fundamental region of the Mod to -Pi to Pi (rather than 0 to 2Pi). The period is taken to be 2 Pi, symmetric around the origin, so the function is even.

\[ ff1[x_] = \text{UnitStep}[\text{Mod}[x + \pi / 2, 2\pi, -\pi]] \text{UnitStep}[\text{Mod}[\pi / 2 - x, 2\pi, -\pi]]; \]

\[ ff1plot = \text{Plot}[ff1[x], \{x, -3\pi, 3\pi\}, \text{Ticks} \rightarrow \{\{-3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi\}\}, \text{PlotStyle} \rightarrow \{\text{Thick, Red}, \text{Exclusions} \rightarrow \text{None}\}] \]

For an even function can use Cosine Fourier Series, here up to Cos[6x]. We will call this a 4 term series.

\[ \begin{align*}
\text{ft6ff1}[x_] &= \text{FourierCosSeries}[ff1[x], x, 6] \\
&= \frac{1}{2} + \frac{2\cos[x]}{\pi} - \frac{2\cos[3x]}{3\pi} + \frac{2\cos[5x]}{5\pi}
\end{align*} \]

The function does a pretty poor job of representing the "barrier" at this order.

\[ \text{Plot}[[ff1[x], \text{ft6ff1}[x]], \{x, -3\pi, 3\pi\}, \text{Ticks} \rightarrow \{\{-3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi\}\}, \text{PlotStyle} \rightarrow \{\{\text{Thick, Red}, \{\text{Thick, Blue}\}\}, \text{Exclusions} \rightarrow \text{None}\}] \]
If one uses "FourierSeries" one gets the complex form, which is equivalent to that found above:

\[
\text{FourierSeries}[ff1[x], x, 6] = \frac{1}{2} + \frac{e^{i x}}{\pi} + \frac{e^{i x}}{3 \pi} - \frac{e^{-3 i x}}{3 \pi} + \frac{e^{-5 i x}}{5 \pi} + \frac{e^{5 i x}}{5 \pi}
\]

Since the number of terms is known, it is faster (for evaluation) to write out the Fourier Transform explicitly.

\[
\text{In[164]} := \text{ftff1[x, _, nmax]} := 1/2 + \text{Sum}[-(-1)^n \text{Cos}[(2 n + 1) x] / (2 n + 1), \{n, 0, \text{nmax}\}] 2 / \text{Pi}
\]

Making and manipulating plots:

\[
\text{In[165]} := \text{barrierplot[n]} := \text{Plot}[[\text{ff1[x]}, \text{ftff1[x, n]}], \{x, -\text{Pi}, \text{Pi}\}, \text{Ticks} \to \{\{-\text{Pi}, -\text{Pi}/2, 0, \text{Pi}/2, \text{Pi}\}\}, \text{PlotStyle} \to \{\{\text{Thick, Red}\}, \{\text{Thick, Blue}\}\}, \text{Exclusions} \to \text{None}]
\]

\[
\text{In[166]} := \text{barrierplot[1]}
\]

\[
\text{Out[166]} :=
\]
In[167]:= `Manipulate`[`barrierplot`[n], {n, 0, 20, 1}]

Comparing the 4, 22 and 102 term Cosine Series to the function itself (which is no longer visible):

Plot[{
  `ff1`[x], `ftff1`[x, 2], `ftff1`[x, 20], `ftff1`[x, 100]
},
{x, -Pi, Pi}, Ticks -> {{-Pi, -Pi/2, 0, Pi/2, Pi}}, PlotStyle ->
  {{Thick, Red}, {Thick, Blue}, {Thick, Green}, {Thick, Black}}, Exclusions -> None]

Zooming in on the edge of the step, showing the 22, 102 and 502 term series.
(Colors match those in the previous plot.)
Note the oscillation near the edge with the fixed 9% overshoot (called the Gibbs phenomenon).
Using Complex Fourier Series

```
In[179]:= ft6ff1complex = FourierSeries[ff1[x], x, 6]
```

```
Out[179]= \[
\frac{1}{2} + \frac{\text{Exp}[-\text{I} x]}{\pi} + \frac{\text{Exp}[\text{I} x]}{\pi} - \frac{\text{Exp}[-3 \text{I} x]}{3 \pi} - \frac{\text{Exp}[3 \text{I} x]}{3 \pi} + \frac{\text{Exp}[-5 \text{I} x]}{5 \pi} + \frac{\text{Exp}[5 \text{I} x]}{5 \pi}
\]
```

```
In[180]:= FullSimplify[ft6ff1complex]
```

```
Out[180]= \[
\frac{15 \pi + 60 \cos(x) - 20 \cos(3 x) + 12 \cos(5 x)}{30 \pi}
\]
```

General real function (neither odd nor even)

```
In[185]:= ff4[x_] := Piecewise[
{\{(3 \text{Mod}[x, 2 \pi] - \pi) / \pi - 1, \text{Mod}[x, 2 \pi] - \pi < -\pi / 2\},
{-\text{Mod}[x, 2 \pi] - \pi / \pi, \text{Mod}[x, 2 \pi] - \pi \geq -\pi / 2\}}
```

```
In[186]:= ff4plot = Plot[ff4[x], \{x, -3 \pi, 3 \pi\}, Ticks \rightarrow \{\{-3 \pi, -2 \pi, -\pi, 0, \pi, 2 \pi, 3 \pi\}\},
PlotStyle \rightarrow \{\text{Thick, Red}\}, \text{Exclusions} \rightarrow \text{None}\]
```

```
Out[186]=
```

Use "FourierTrigSeries" to get Cosine/Sine series
Here is the result up to 6th order

```
ln[188]= ft6ff4[x_] = FourierTrigSeries[ff4[x], x, 6]
Out[188]= 1/4 (1 + 4 Cos[x]/\pi^2 - 2 Cos[2 x]/\pi^2 + 4 Cos[3 x]/\pi^2 + 4 Cos[5 x]/25 \pi^2 - 9 \pi^2/\pi^2 + 9 \pi^2 + 9 \pi^2 - 25 \pi^2)
```

```
ln[189]= Plot[{ff4[x], ft6ff4[x]}, {x, -3 Pi, 3 Pi},
          Ticks -> {(-3 Pi, -2 Pi, -Pi, 0, Pi, 2 Pi, 3 Pi)},
          PlotStyle -> {Thick, Red}, {Thick, Blue}], Exclusions -> None]
```

Here's the complex version:

```
In[190]= ft6ff4complex = FourierSeries[ff4[x], x, 6]
Out[190]= \frac{1}{\pi^2} \left[ 1 + \frac{(2 - 2 i) e^{-ix}}{\pi^2} + \frac{(2 + 2 i) e^{ix}}{\pi^2} - \frac{e^{-2 i x}}{\pi^2} - \frac{e^{2 i x}}{\pi^2} + \frac{\left(\frac{2}{9} + \frac{2 i}{9}\right) e^{-3 i x}}{\pi^2} + \frac{\left(\frac{2}{25} - \frac{2 i}{25}\right) e^{-5 i x}}{\pi^2} + \frac{\left(\frac{2}{25} + \frac{2 i}{25}\right) e^{5 i x}}{\pi^2} + \frac{e^{-6 i x}}{\pi^2} - \frac{e^{6 i x}}{\pi^2} \right] + \left(\frac{2}{9} - \frac{2 i}{9}\right) e^{-3 i x} + \left(\frac{2}{25} - \frac{2 i}{25}\right) e^{-5 i x} + \left(\frac{2}{25} + \frac{2 i}{25}\right) e^{5 i x} + \frac{e^{-6 i x}}{\pi^2} - \frac{e^{6 i x}}{\pi^2}
```

```
In[191]= FullSimplify[ft6ff4complex]
Out[191]= \frac{1}{900 \pi^2} \left( 3600 \cos[x] - 1800 \cos[2 x] + 400 \cos[3 x] + 144 \cos[5 x] - 25 \left( 9 \pi^2 + 8 \cos[6 x] + 144 \sin[x] - 16 \sin[3 x] \right) - 144 \sin[5 x] \right)
```

Even function with discontinuity in derivative

Here we consider an example with a discontinuity in derivative but not in the function itself.

```
ff3[x_] = Piecewise[{{\cos[x], -\pi/2 <= Mod[x, 2 \pi, -\pi] < \pi/2}}]
```

```
Out[192]= \left\{ \begin{array}{ll}
\cos[x] & -\frac{\pi}{2} \leq \text{Mod}[x, 2 \pi, -\pi] < \frac{\pi}{2} \\
0 & \text{True}
\end{array} \right.
```
Starting with the \( \cos(2x) \) term, the number in the denominator is \( n^2-1 \).
So the coefficients fall like \( 1/n^2 \), compared to the \( 1/n \) we saw with a discontinuity in the function itself.
This is general.

\[
\begin{align*}
ft6ff3[x_] &= \text{FourierCosSeries}[ff3[x], x, 6] \\
&= \frac{1}{\pi} \cos(x) + \frac{2}{2} \cos(2x) + \frac{2}{3} \cos(4x) + \frac{2}{15} \cos(6x) + \frac{2}{35} \cos(8x) + \frac{2}{63} \cos(10x) \\
ft20ff3[x_] &= \text{FourierCosSeries}[ff3[x], x, 20] \\
&= \frac{1}{\pi} \cos(x) + \frac{2}{3} \cos(2x) + \frac{2}{15} \cos(4x) + \frac{2}{35} \cos(6x) + \frac{2}{63} \cos(8x) + \frac{2}{99} \cos(10x) \\
&\quad+ \frac{2}{143} \cos(12x) + \frac{2}{195} \cos(14x) + \frac{2}{255} \cos(16x) + \frac{2}{323} \cos(18x) + \frac{2}{399} \cos(20x)
\end{align*}
\]

\[
\begin{align*}
ft100ff3[x_] &= \text{FourierCosSeries}[ff3[x], x, 100];
\end{align*}
\]

From this view, 20 terms is enough to do a good job, with 6 showing oscillations:

\[
\begin{align*}
\text{fft3plot} &= \text{Plot}[\{ff3[x], ft6ff3[x], ft20ff3[x]\}, \\
&\quad\{x, -3\pi, 3\pi\}, \text{Ticks} \to \{\{-3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi\}\}, \\
&\quad\text{PlotStyle} \to \{\text{Thick, Red}, \{\text{Thick, Blue}, \{\text{Thick, Green}\}\}, \text{Exclusions} \to \text{None}]}
\end{align*}
\]
Zooming in, we need > 100 terms to approach the discontinuity in derivative. But there is no Gibbs phenomenon here.

\[
\text{ff3plot} = \text{Plot}[[\text{ff3}[x], \text{ft6ff3}[x], \text{ft20ff3}[x], \text{ft100ff3}[x]], \\
\{x, \Pi/2 - 0.1, \Pi/2 + 0.2\}, \text{PlotStyle} \rightarrow \\
\{\{\text{Thick, Red}\}, \{\text{Thick, Blue}\}, \{\text{Thick, Green}\}, \{\text{Thick, Black}\}\}, \text{Exclusions} \rightarrow \text{None}]
\]

**Integrals needed in discussion of Gibbs phenomenon (see lecture notes)**

\[
\text{Integrate}[\sin(x)/x, \{x, 0, \text{Infinity}\}] / \Pi
\]

\[
\frac{1}{2}
\]

\[
\text{Integrate}[\sin(x)/x, \{x, 0, \Pi\}] / \Pi // \text{N}
\]

\[
0.58949
\]

Showing that the sum indeed asymptotes to the integral above:

\[
\text{ss}[N] := \text{Sum}[\sin[(2n + 1)\Pi/(2(N + 1))] / (2n + 1), \\
\{n, 0, N\}, \text{Assumptions} \rightarrow \{N \in \text{Integers}, N > 0\}] 2 / \Pi
\]

\[
\text{ListPlot}[[\text{Table}[ss[n], \{n, 5, 500\}]]]
\]
Fourier transforms

Section 7.12 problem number 10

This is one way of writing the function in Mathematica

\[ f_{10}(t) = \text{Piecewise} \left\{ \begin{array}{ll} -2(a + t), & -a \leq t < 0 \\ 2(a - t), & 0 < t < a \end{array} \right\} \]

Plotting: note that the "/. a->1" command tells Mathematica to set the parameter \( a \) to 1 in the function.

\[ f_{10}\text{plot} = \text{Plot}[f_{10}(t) /. a \to 1, \{t, -2, 2\}, \text{PlotStyle} \to \{\text{Thick}, \text{Red}\}] \]

Obtaining the Fourier Transform.
The "FourierParameters" must be set to match our conventions (see the Documentation for their definitions).
The "Assumptions -> a>0" allows Mathematica to ignore the case where \( a<0 \) in which case, given the definitions above, the function vanishes.
"FullSimplify" leads to a more compact answer than "Simplify"

\[ f_{10}(\omega) = \text{FullSimplify} \left[ \text{FourierTransform} \left[ f_{10}(t), t, \omega, \text{FourierParameters} \to \{-1, -1\}, \text{Assumptions} \to a > 0 \right] \right] \]

Since our function is odd, we can also use the "FourierSinTransform", which gives the imaginary part of the previous result

\[ \text{img}_{10}(\omega) = \text{FourierSinTransform} \left[ f_{10}(t), t, \omega, \text{FourierParameters} \to \{-1, -1\}, \text{Assumptions} \to a > 0 \right] \]
Here is the inverse Fourier Transform which brings us back to the original function (expressed in a different way).

In[212]:= invft10[t_] = FullSimplify[
    InverseFourierTransform[ft10[alp], alp, t, FourierParameters \[RightTilde] {-1, -1}]]

Out[212]= (a-t) \[CapitalDelta][a-t] + 2a \[CapitalDelta][t] - (a+t) \[CapitalDelta][a+t]

In[213]:= Plot[invft10[t] /. a \[Rule] 1, {t, -2, 2}, PlotStyle \[RightTilde] {Thick, Red}]