Testing for convergence (continued): \( S = \sum_{n=1}^{\infty} a_n \)

- Assume \( S \) passes prelim. test, i.e. \( \lim_{n \to \infty} a_n = 0 \)

  e.g. \( S_{\text{harm}} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \quad \text{harmonic series} \)

  or \( S_{\text{alt-harm}} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \quad \text{alternating harmonic series} \)

Aside: \( S \) may have terms of both signs,

  e.g. \( S_{\text{alt-harm}} \)  

  If we consider, instead of \( S \), the related series \( \tilde{S} = \sum_{n=1}^{\infty} |a_n| \), then

  \( \tilde{S} \) is less convergent than \( S \) (since no cancellation)

  \( \Rightarrow \) If \( \tilde{S} \) converges, then so does \( S \) (N.B. Converse is not true)

  Challenge: make this argument watertight.

  - In this case we say that \( S \) is "absolutely convergent," which is stronger than convergent.

  - Clearly if all \( a_n \) have the same sign, then \( \tilde{S} = \pm S \) (convergence = abs. conv.)

  - For the moment, we'll study absolute convergence, or, equivalently, consider series w/ all coefficients having the same sign.
Comparison test

(a) Let $C = \sum_{n=1}^{\infty} c_n$, with $c_n > 0$, be a convergent series.

Then, if $0 \leq a_n \leq c_n$ for all $n \geq N$,

$$S = \sum_{n=1}^{\infty} a_n$$

converges.

(b) If $D = \sum_{n} d_n$, with $d_n > 0$, is divergent, and $a_n \geq d_n$ for all $n \geq N$, then $S$ diverges.

Why? In case (a) $0 < \sum_{n=N}^{\infty} a_n \leq \sum_{n=N}^{\infty} c_n < \infty$

[Strictly speaking—need "monotone convergence theorem"][1]

So $S = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n = \text{finite, so convergent}$

$\text{manifestly finite}$

$\text{finite from above}$

Similarly for (b).

Comparison test useful when have a catalog of series whose convergence or divergence is known.
Examples:

§1.6 # 4(a)

\[ S = \sum_{n=1}^{\infty} \frac{1}{2^n + 3^n} \quad \text{Geometric,} \quad r = \frac{1}{3} < 1 \Rightarrow \text{convergent} \]

\[ C = \sum_{n=1}^{\infty} \frac{1}{3^n} \quad \text{convergent} \]

\[ 0 < a_n < c_n \quad \forall n \Rightarrow S \text{ convergent} \]

"for all"

\[ \sqrt[n]{a_n} \]

§1.6 # 5

\[ S = \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots \]

\[ D = \sum_{n=1}^{\infty} \frac{1}{n} = \text{harmonic series} - \text{will see} \]

we will see divergent soon

\[ a_n \geq d_n \quad \forall n \Rightarrow S \text{ divergent} \]
Integral test

\[ S = \sum_{n=1}^{\infty} a_n \]

**Conditions**
- \( a_n \geq a_{n+1} \geq 0 \quad \forall \ n \geq N \)
  - i.e. positive, monotonically decreasing terms at large \( n \)
- Need an expression for \( a_n \) valid for all real \( n \)
  - Not just integers
  - E.g. \( \frac{1}{n^2} \), \( \frac{1}{n\ln(n)} \), \( \frac{1}{n} \)

**Results**
- If \( \int_{a}^{\infty} a_n \, dn \) is finite at upper limit
  - Then \( S \) converges
- If \( \int_{a}^{\infty} a_n \, dn \) diverges at upper limit
  - Then \( S \) diverges

**Examples**
- Harmonic series: \( a_n = \frac{1}{n} \)
  - \( \int_{1}^{\infty} \frac{1}{n} \, dn = \left[ \ln(n) \right]_{1}^{\infty} = \infty \)
  - \( \Rightarrow \) diverges
- \( a_n = \frac{1}{n^2} \)
  - \( \int_{1}^{\infty} \frac{1}{n^2} \, dn = \left[ -\frac{1}{n} \right]_{1}^{\infty} = 0 \)
  - \( \Rightarrow \) converges.

**N.B.** Integral test is powerful, but we must be able to do the integral.
Why does integral test work?  See Boas Figs 6.1 & 6.2

e.g.

\[ \sum_{n=N}^{\infty} a_n \text{ given by sum of areas of rectangles} \]

\[ \int_{n=N}^{\infty} a_n \, dn \text{ given by red shaded area} \]

Clearly \[ \sum_{n=N}^{\infty} a_n > \int_{N}^{\infty} a_n \, dn \]

so if \[ \int_{N}^{\infty} a_n \, dn = \infty \] then sum diverges

(Recall \[ \sum_{n=1}^{N} a_n \] is always finite – divergence only occurs when consider \( \infty \) # of terms.)
Ratio test: weaker than integral test, but often useful in practice

Let \( p_n = \left| \frac{a_{n+1}}{a_n} \right| \) be the ratio of successive terms

and assume \( p = \lim_{n \to \infty} p_n \) exists

Then, if \( p < 1 \) \( \sum a_n \) converges absolutely

while if \( p > 1 \) \( S \) diverges

If \( p = 1 \), ratio test does not give an answer

Examples:

- \( a_n = \frac{1}{n!} \) (hard to integrate) \( \Rightarrow p_n = \frac{n!}{(n+1)!} = \frac{1}{n+1} \Rightarrow p = 0 \)

  \( \Rightarrow \) convergent

- \( \#1.6 \) \#24 \( a_n = \frac{3^{2n}}{2^n} \) \( S = \sum_{n=0}^{\infty} a_n = 1 + \frac{9}{8} + \frac{81}{64} + \ldots \)

  Looks divergent... let's check w/ ratio test
  (could use prelude test too).

  \( p_n = \frac{3^2}{2^3} = \frac{9}{8} = p > 1 \Rightarrow \) divergent
Sketch of proof of ratio test (see Boas §1.6 #30)

if \( \rho < 1 \)

- Let \( \sigma \) be such that \( \rho < \sigma < 1 \)

- \( \exists N \) ("there exists an \( N \)) s.t. \( p_n < \sigma \ \forall n \geq N \)

  - e.g. \( \frac{p_n}{p_{n+1}} \) vs. \( \rho \)

  - \( \Rightarrow \ |a_{N+1}| < \sigma |a_N| \), \( |a_{N+2}| < \sigma |a_{N+1}| \), etc.,

- Now consider convergent geometric series

  \[
  C = \sum_{n=1}^{\infty} \frac{|a_n|}{\sigma^n} \quad \text{for which } p_n = \sigma
  \]

- We observe \( c_N = |a_N| \), \( c_{N+1} = \sigma |a_N| > |a_{N+1}| \), \( c_{N+2} > |a_{N+2}| \), etc.,

- Use comparison test

  \[
  \Rightarrow S = \sum_n |a_n| \text{ is convergent}
  \]

  \[
  \Rightarrow S = \sum_n a_n \text{ converges absolutely.}
  \]

[Similar argument works for \( \rho > 1 \)]
"Combined" or "Special comparison" test for $S = \sum_{n=1}^{\infty} a_n$

small, but important, extension of comparison test

(a) If $C = \sum_{n=1}^{\infty} c_n$ converges (with $c_n > 0$),
and $\lim_{n \to \infty} \frac{|a_n|}{c_n} < \infty$, then $S$ converges.

(b) If $D = \sum_{n=1}^{\infty} d_n$ diverges (with $d_n > 0$),
and $\lim_{n \to \infty} \frac{|a_n|}{d_n} > 0$, then $S$ diverges.

Key point: only terms at large $n$ matter.

Proof: see §1.6 #37

Ex: §1.6 #36 $a_n = \frac{\sqrt{n^3 + 5n - 1}}{n^2 - \sin(n^3)}$ Weird!

Note $a_n \xrightarrow[n \to \infty]{} \frac{\sqrt{n^3}}{n^2} = \frac{1}{\sqrt{n}}$

So if $d_n = \frac{1}{\sqrt{n}}$ then $\lim_{n \to \infty} \frac{a_n}{d_n} = 1 > 0$ for $n \geq 1$.

Since $\Sigma d_n$ diverges (see above) we learn that $S$ diverges.