Continuing with example: \( \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \)

For what range of \( x \) does series converge?

**Preliminary:** \( |a_n| \to \infty \) if \( |x| > 1 \)

**Ratio:** \( \rho_n = \left| \frac{a_{n+1}}{a_n} \right| = \frac{x^n}{n+1} \to x = \rho \)

\( \Rightarrow \) absolutely convergent for \( |x| < 1 \)

Boundary cases?

\( x = 1 \):
\( 1 - \frac{1}{2} + \frac{1}{3} \ldots \) alternating harmonic convergent by alternating test

\( x = -1 \):
\(-1 + \frac{1}{2} + \frac{1}{3} \ldots \) harmonic \( \Rightarrow \) divergent

**Summary:** Range of convergence is \([-1 < x \leq 1]\) (a.k.a. "interval of convergence")

N.B. Always get symmetric interval aside from boundary cases

- What is function? Taylor series for \( \log(1+x) \)

- Can now sum alternating harmonic series (set \( x = 1 \))

  \( 1 - \frac{1}{2} + \frac{1}{3} \ldots = \log(2) \)

Example of using Taylor to sum series.

\( \text{but not at } x \to +1 \)
Another example

Boas (10.2d) \[ b_n = \frac{(x+2)^n}{\sqrt{n+1}} \] (starts at \( n=0 \))

Series is \[ 1 + \frac{x+2}{\sqrt{2}} + \frac{(x+2)^2}{\sqrt{3}} + \ldots \]

Easiest to change variable to \( y = x+2 \).

* Prelim test: diverges for \( |y| > 1 \)

* Ratio test: \( \rho_n = y \frac{\sqrt{n+1}}{\sqrt{n+2}} \xrightarrow{n \to \infty} y \)

abs. conv. for \(|y| < 1\)

* \( y = +1 \): \( 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots \) diverges by comparison with harmonic series.

* \( y = -1 \): \( 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots \) converges by alternating series test.

\[ \Rightarrow \text{convergent for } -1 \leq y < 1 \]

Convert to \( x \):

\[ -3 \leq x = (y-2) < -1 \]

Aside: The series defines a function called the "Lerch transcendent" function.

Aside from end points

Symmetrical interval used if use correct variable
Final example

\[ b_{2n} = \left( \frac{(-1)^n}{(2n)!} \right) \quad b_{2n+1} = 0 \]

i.e. \[ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \] familiar?

Preliminary test: note that \( n! \) grows faster than \( x^n \)

\[ \text{Stirling's formula:} \quad \ln (n!) = n \ln n - n + O(\ln(n)) \]
\[ \approx n \ln(n/e) \]
\[ \approx \ln[(n/e)^n] \]

Alternatively (including high-order term)

\[ n! \sim \left( \frac{n}{e} \right)^n \sqrt{2\pi n} \]

so passes prelim test for all \( x \).

Ratio test: apply to contiguous terms (or treat as power series in \( x^2 = y \))

\[ \sqrt{n} = \frac{x^2}{(n+1)(n+2)} \quad n \to \infty \]

\[ \Rightarrow \quad \text{abs. conv. for all } x \]

This function is \( \cos(x) \)

- periodically NOT manifest in power series.
Key results about power series & their convergence (stated w/o proofs) Boas 1.11

* Let \( S(x) = \sum_{n=0}^{\infty} a_n x^n \) define a function in the interval of convergence of the series

Then

* The power series is **unique** - only one choice of \( a_n \) works
  \[ \Rightarrow \text{Taylors series completely defines a function in its interval of convergence} \]

* Power series can be integrated & differentiated term by term leaving range of convergence unchanged (although properties at end points may change)

I.e., \( S'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1} \leq \text{series converges in some range of } S \) and converges to \( S' \)

\[ \int S(y) \, dy = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} \]

* Can add, subtract & multiply series - if both are convergent - and also divide in some cases (see Boas for more discussion)

* Can substitute one series into another - with some conditions

Best to understand via examples.

N.B. Mathematica can do the tedious work for us, but we need to understand what it is doing.
Need some basic examples to start with — can obtain from Taylor series

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \]

converges for all \( x \) (\( p = \frac{|x|}{n+1} \))

\[ \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots \] (seen above)

convergent for \(-1 < x \leq 1\)

\[ \frac{1}{1-x} = x + x^2 + x^3 + \ldots \] (geometric series seen above)

convergent for \(-1 < x < 1\)

\[ (1+x)^p = 1 + px + \frac{p(p-1)x^2}{2} + \ldots \] [binomial expansion]

\[ = \sum_{n=0}^{\infty} \binom{p}{n} x^n \]

\[ \text{binomial coeff } \binom{p}{n} = \frac{p(p-1)\ldots(p-n+1)}{n!} \]

* valid for any \( p \)

* convergent for \(-1 < x < 1\)

N.B. Functions defined for wider range than that for which series converges.

Read Boas §10.2 if not comfortable obtaining these Taylor series.
I'm assuming you know how to develop these expressions, but let's do one for practice.

\[ f(x) = \ln(1+x) \Rightarrow f(0) = 0 \]

\[ f'(x) = \frac{1}{1+x} \quad \Rightarrow \quad f'(0) = 1 \]

\[ f''(x) = \frac{-1}{(1+x)^2} \quad \Rightarrow \quad f''(0) = -1 \]

\[ f'''(x) = \frac{2}{(1+x)^3} \quad \Rightarrow \quad f'''(0) = 2 \]

\[ f''''(x) = \frac{-6}{(1+x)^4} \quad \Rightarrow \quad f''''(0) = -6 \]

Taylor series:

\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \]

\[ = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots \]
Example of integration to obtain a series:

Use \( \int_0^x \frac{dy}{1+y^2} = \tan^{-1} x \) to determine power series for \( \tan^{-1} x \).

\[
\tan^{-1} x \text{ is not unique -- integral gives this branch.}
\]

**even term**

\[
\frac{1}{1+y^2} = 1 - y^2 + y^4 - y^6 + \ldots \quad \text{geometric convergent for } 0 < y^2 < 1.
\]

**odd term**

\[
\int_0^x \frac{1}{1+y^2} = x - x^3 + x^5 - x^7 + \ldots = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}
\]

abs convergent? ratio test \( p_n = \frac{x^2 (2n+1)}{(2n+3) n!} \to \frac{x^2}{n^3} \to 0 \) so abs. convergent for \( 0 < x^2 < 1 \)

\( x^2 = 1 \) — conditionally convergent

\( \Rightarrow \) series for \( \tan^{-1} x \) converges for \( 0 < x^2 \leq 1 \)

N.B. Same interval as \( \int \frac{1}{1+y^2} \) except now have "gained" end point.
Example of multiplying series

\[ e^x \cos x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) \times \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right) \]

\[ = 1 + x + \frac{x^2}{2} - \frac{x^2}{2} = 1 + x + 0 \]

\[ + \frac{x^3}{2} - \frac{x^3}{2} + \frac{x^3}{6} \]

\[ + \frac{x^4}{24} - \frac{x^2}{4} + \frac{x^4}{24} - \frac{x^4}{6} \]

Clearly, to get to a given order, need to expand each series to this order first (in general).

Very tedious - good task for computer.

Dividing series - even more tedious! (can use long division, see Boas p36).

Simple example

\[ \frac{\sin x}{x} = \left(x - \frac{x^3}{6\pi} + \frac{x^5}{5\pi^5} + \cdots\right) \]

\[ = 1 - \frac{x^2}{3\pi} + \frac{x^4}{5\pi^5} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2\pi n)!} \]

Abs. Convergent for all \( x \in \mathbb{R} \) (just like \( \sin x \)).