1. [10 pts total] Lagrangian mechanics. A particle of mass $m$ moves in three dimensions and is described using cylindrical coordinates $\{r, \theta, z\}$. It is subject to a potential $V = \alpha r^4 + \beta z^2$. Determine the equations of motion, i.e. find the differential equations that must be solved to find $r(t), \theta(t)$ and $z(t)$. You do not need to solve these equations. However, you should give an expression for one conserved quantity other than the total energy, and state its physical interpretation. (A conserved quantity is a quantity that is independent of time.)

$$L = T - V \quad ; \quad V = \alpha r^4 + \beta z^2$$

$$T = \frac{1}{2} m r^2 = \frac{1}{2} m \left( r^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right)$$

$$\Rightarrow L = \frac{m}{2} \left( r^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right) - \alpha r^4 - \beta z^2.$$

Euler Lagrange eqs (eqs of motion):

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \Rightarrow \frac{d}{dt} m \ddot{r} = m \dot{r}^2 + mr \ddot{\theta} - 4\alpha r^3$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \Rightarrow \frac{d}{dt} (mr^2 \dot{\theta}) = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{z}} = \frac{\partial L}{\partial z} \Rightarrow m \ddot{z} = -2\beta \dot{z}$$

From $\theta$ equation we learn the first integral

$$mr^2 \dot{\theta} = \text{constant}.$$

This is the angular momentum about the $z$-axis.
2. [10 pts] Gamma and Beta functions.

(a) It is known that $\Gamma(1.5) = \sqrt{\pi}/2$. Find the value of $\Gamma(-1.5)$.

(b) Express the following integral in terms of Gamma and/or Beta functions:

$$ I = \int_0^\infty x^\frac{1}{2} e^{-x^3} dx. $$

You do not need to evaluate it numerically.

(a) Use $\Gamma(p) = \frac{\Gamma(p+1)}{p}$, so $\Gamma(-1.5) = \frac{\Gamma(-0.5)}{-1.5} = \frac{\Gamma(0.5)}{(-1.5)\Gamma(-0.5)}$.

\[ \Rightarrow \Gamma(-1.5) = \frac{8}{3} \Gamma(1.5) = \frac{4}{3} \sqrt{\pi}. \]

(b) $I = \int_0^\infty x^\frac{1}{2} e^{-x^3} dx$.

Use $x^3 = u$, $du = 3x^2 dx$.

$$ I = \int_0^\infty \frac{1}{3} x^{-\frac{3}{2}} e^{-u} du = \int_0^\infty \frac{1}{3} u^{-\frac{1}{2}} e^{-u} du $$

In form needed for Gamma function.
3. [1] pts total] Legendre Series. Determine the coefficients $c_n$ in the Legendre series

$$P_n'(x) = \frac{dP_n(x)}{dx} = \sum_{n=0}^{\infty} c_n P_n(x).$$

Hint: First use general arguments to determine which coefficients are non-zero, then calculate the non-zero coefficients (using integration by parts).

$P_5(x)$ is a 5th order polynomial

$\Rightarrow P_5'$ is an even 4th order polynomial

$\Rightarrow$ only $c_0$, $c_2$, & $c_4$ are non-zero

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} P_n(x) P_5'(x) \, dx$$

$$= \frac{2n+1}{2} \left[ \int_{-1}^{1} P_n(x) P_5(x) \right] \bigg|_{-1}^{1} - \frac{2n+1}{2} \int_{-1}^{1} P_n(x) P_5(x) \, dx$$

Integrate by parts:

$$1 - (-1) = 2$$

since $P_0(x) = 1$, $P_2(x) = 1$, $P_4(x) = 1$

Polynomial of order $n-1$

i.e. vanishes for $n=0$

linear for $n=2$, cubic for $n=3$.

This integral vanishes in all 3 cases from orthogonality of Legendre.

$\Rightarrow c_n = 2n+1$ for $n=0, 2, 4$

$\Rightarrow c_0 = 1, c_2 = 5, c_4 = 9$
4. **14 pts total** PDEs and separation of variables. We consider here the temperature distribution inside a long, narrow rod of length $L$ with insulated sides, which we can solve by treating it as a 1-D diffusion problem. If $x$ is the coordinate along the rod (running from 0 to $L$), then the temperature $T(x,t)$ satisfies

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial T}{\partial t}. \quad (\star)$$

For $t < 0$, the slab is in steady-state equilibrium, with the $x = 0$ end held at 0° while the other end is held at temperature $T_0$. At $t = 0$, and henceforward, both ends are held at 0°.

(a) **4 pts** Separate variables $T = X(x)\tau(t)$ and write down the differential equations satisfied by the $x$ and $t$-dependent functions.

Substitute into $(\star)$ & obtain

$$X'' \tau = \frac{1}{\alpha^2} \tau \Rightarrow \frac{X''}{X} = \frac{1}{\alpha^2} \frac{\tau}{\tau} = -\alpha^{-2} \Rightarrow \tau = -(\alpha \cdot \alpha)^2 \tau$$

$$\Rightarrow \begin{cases} X'' = -\alpha^{-2} X \\ \tau = -(\alpha \cdot \alpha)^2 \tau \end{cases}$$

(b) **4 pts** Solve for $T(x,t)$ for $t < 0$.

For $t < 0$, $\tau$ is a constant, since $\dot{T} = 0$. Choose $\tau = 1$.

Then $\alpha^2 \tau = 0 \Rightarrow X'' = 0$

$$\Rightarrow X(x) = a + bx \quad \text{linear form}$$

BC: $X(0) = 0$; $X(L) = T_0$

$$\Rightarrow X(x) = \frac{T_0 x}{L} = T(x,t) \text{ for } t < 0$$
(c) [6 pts] Write down the general form of the solution \( T(x, t) \) for \( t > 0 \), incorporating the boundary conditions in \( x \) and that for \( t \to \infty \). This form will contain undetermined constants.

For \( t > 0 \), B.C. in \( x \) is \( X(0) = X(L) = 0 \)

Solutions of \( X'' = -k^2 X \) are \( \sin kx \) & \( \cos kx \).

(we see now why negative sep. constant is needed)

B.C. \( \Rightarrow X_n' = \sin \frac{n\pi x}{L} \quad n=1, 2, 3, ... \)

\[ \Rightarrow k = \frac{n\pi}{L} \]

Thus eq. is \( \dot{Z} = -\left(\frac{n\pi}{L}\right)^2 Z \)

solutions \( Z_n(t) = e^{-\left(\frac{n\pi}{L}\right)^2 t} \)

(choosing \( T_n(0) = 1 \) for convenience)

General solution

\[ T(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 t} \]
(d) [4 pts] Derive a formula for the undetermined constants in your expression. You do not need to explicitly evaluate any integrals.

\[ T(x, 0) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n \pi x}{L} \right) = T_0 \frac{x}{L} \]

To determine \( c_m \), integrate with \( \sin \left( \frac{m \pi x}{L} \right) \)

\[ \xi = \int_0^L \sin \left( \frac{m \pi x}{L} \right) \sin \left( \frac{n \pi x}{L} \right) dx = \delta_{mn} \frac{L}{2} \]

\[ \Rightarrow \quad c_m \frac{L}{2} = \frac{T_0}{L} \int_0^L \sin \left( \frac{m \pi x}{L} \right) \times dx \]

or

\[ c_m = \frac{2T_0}{L^2} \int_0^L \sin \left( \frac{m \pi x}{L} \right) \times dx \]