Series solutions. Consider the following differential equation for $y(x)$

$$(2 - x^2)y'' + 2xy' - 2y = 0.$$ 

We want to find the general solution (which will depend on two constants) using the method of series, i.e. with the form $y = \sum_{n=0}^{\infty} a_n x^n$.

1. (10 pts) Determine the recursion relation between the coefficients $a_n$.

\[
xy' = \sum_{n=0}^{\infty} n a_n x^n \quad x^2 y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^n \\
y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n
\]

Combining
\[
\sum_{n=0}^{\infty} x^n \left[ 2(n+2)(n+1) a_{n+2} - n(n-1) a_n + 2n a_n - 2 a_n \right] = 0
\]

must vanish

\[
\Rightarrow 2(n+2)(n+1) a_{n+2} = (n(n-1) - 2n + 2) a_n = (n^2 - 3n + 2) a_n = (n-2)(n-1) a_n
\]

or (equally good)
\[
a_{n+2} = \frac{(n-2)(n-1)}{2(n+2)(n+1)} a_n
\]
2. (10 pts) Solve the recursion relation to find the general solution, which you can leave in the form of a power series. Hint: The series truncates after a finite number of terms.

Recursion relation connects even to even and odd to odd terms.

- Start with $n=0$
  \[ a_2 = \frac{\frac{3}{4}}{2} a_0 = \frac{a_0}{2} \quad ; \quad a_4 = \emptyset \Rightarrow a_6 = a_8 = \ldots = \emptyset \]
  so another solution is $a_0 \left( 1 + \frac{x^2}{2} \right)$

- Start with $n=1$
  \[ a_3 = 0 \Rightarrow a_5 = a_7 = \ldots = \emptyset \]
  so another solution is $a_1 x$

General solution:
\[ a_0 \left( 1 + \frac{x^2}{2} \right) + a_1 x \]