 Physiology 228: First Midterm—useful results that may be used without explanation

**General linear first-order ODE:**

\[
y'(x) + P(x)y(x) = Q(x) \Rightarrow y = e^{-\int P \, dx} \int Q \, e^\int P \, dx + c e^{-\int P \, dx} \quad I = \int P \, dx.
\]

**Linear homogeneous second-order ODE with constant coefficients:**

\[
(D - \lambda_1)(D - \lambda_2)y(t) = f(t), \quad [D = \frac{d}{dt}],
\]

has general solution \(y(t) = y_C(t) + y_P(t)\) where the complementary solution is

\[y_C(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},\]

(if \(\lambda_1 \neq \lambda_2\)) and \(y_P(t)\) is a particular solution.

**Analytic functions** satisfy Cauchy-Riemann conditions \((f(z) = u + iv)\):

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.
\]

**Residue theorem** for closed, counterclockwise contour

\[
\oint_C f(z)dz = 2\pi i \times \text{sum of residues inside } C.
\]

Residue of single pole at \(z = a\) is \(\lim_{z \to a}(z - a)f(z)\).

If \(f(z) = g(z)/h(z)\) with \(h(z)\) vanishing at \(z = a\), residue is \(g(a)/h'(a)\), assuming \(h'(a) \neq 0\).

At an \(n\)'th-order pole, residue is, for any \(m \geq n\),

\[
\frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} [(z - a)^m f(z)].
\]

For a pole lying on the contour, one obtains half the residue using Cauchy’s principal value prescription.

**Taylor Series.** \(e^x\), \(\cos x\) and \(\sin x\) have no poles in the complex plane, and convergent series representations:

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots, \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots.
\]

**Jordan’s Lemma:** \(\lim_{R \to \infty} \oint_C e^{iz} f(z)dz = 0\) where \(z = Re^{i\theta}, \theta\) runs from 0 to \(\pi\), and the result holds if \(f(z) \to 0\) along the entire semicircular contour.
Laplace transforms.  \([u(t)\text{ is the Heaviside (step) function.}]\]
\[
L[f] = F(p) = \int_0^\infty e^{-pt}f(t)dt,
\]
\[
L[\dot{f}] = pL[f] - f(0), \quad L[\ddot{f}] = p^2L[f] - pf(0) - \dot{f}(0),
\]
\[
L[e^{at}] = \frac{1}{p-a}, \quad L[t^k] = \frac{k!}{p^{k+1}}, \quad L[t^ke^{at}] = \frac{k!}{(p-a)^{k+1}},
\]
\[
L[\sin at] = \frac{a}{p^2 + a^2}, \quad L[\cos at] = \frac{p}{p^2 + a^2},
\]
\[
L[at - \sin at] = \frac{a^3}{p^2(p^2 + a^2)}, \quad L[\sin at - at \cos at] = \frac{2a^3}{(p^2 + a^2)^2},
\]
\[
L[\delta(t - a)] = e^{-pa} \quad (a > 0), \quad L[u(t - a)g(t - a)] = e^{-pa}L[g(t)] \quad (a > 0),
\]
\[
L[g \ast h] = L[g]L[h] \quad \text{where \quad} (g \ast h)(t) = \int_0^t g(\tau)h(t - \tau)d\tau.
\]

Inverse Laplace transform (Bromwich integral):
\[
f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(p)e^{pt}dp = \text{sum of residues of} \ F(p)e^{pt} \text{at all poles}
\]
where \(c > k\), with \(k\) the real part of the “rightmost” singularity in \(F(p)\). The result in terms of residues holds if the contour can be closed to the left.

Dirac delta-function:
\[
\int f(t) \delta(t - t_0) \, dt = \begin{cases} f(t_0) & \text{if integration range includes } t_0 \\ 0 & \text{otherwise} \end{cases}
\]

Changing variables with a delta-function when \(f(x_i) = 0\)
\[
\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}.
\]

Derivatives of delta functions:
\[
\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x - a)dx = (-1)^n f^{(n)}(a).
\]

Green function: Useful for linear ODEs. For example, if \(G(x, x')\) solves
\[
\frac{\partial^2}{\partial^2x}G(x, x') + b \frac{\partial}{\partial x}G(x, x') + cG(x, x') = \delta(x - x') ,
\]
then
\[
y(x) = \int G(x, x')f(x')dx' \quad \text{solves} \quad y''(x) + by'(x) + cy(x) = f(x),
\]
as long as \(y\) and \(G\) satisfy the same homogeneous boundary conditions.