Euler equation. Extremizing \( I[y] = \int_{x_1}^{x_2} F(x,y,y') \, dx \) over paths \( y(x) \) with fixed endpoints \( \Rightarrow \)

\[
\frac{d}{dx} \frac{\partial F}{\partial y'} = \frac{\partial F}{\partial y} .
\]

If \( F \) is independent of \( x \), then \( F - y' \frac{\partial F}{\partial y'} \) is constant.

To extremize \( I \) subject to \( J[y] = c \), use Lagrange multiplier and extremize \( I + \lambda J \).

Euler-Lagrange equation(s). Extremizing action \( S \[\vec{y}(t)\] = \int L(t,\vec{y},\vec{y}') \, dt \), with \( L = T - V \), \( \Rightarrow \) equations of motion are

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{y}_j} = \frac{\partial L}{\partial y_j} \quad (j = 1, 2, 3).
\]

Cylindrical coordinates: \( \{r, \theta, z\} \) with \( x = r \cos \theta, \ y = r \sin \theta \) \( \Rightarrow \)

\[
\vec{v}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2 .
\]

Frobenius’ method (includes series solution). Solve an ODE using

\[
y(x) = x^s \sum_{n=0}^{\infty} a_n x^n \quad \text{with} \quad a_0 \neq 0 .
\]

Guaranteed to give at least one solution of ODE \( y'' + f(x)y' + g(x)y = 0 \), if \( xf(x) \) and \( x^2g(x) \) have Taylor expansions about \( x = 0 \).

Legendre’s equation and Polynomials.

\[
(1-x^2)P''_\ell - 2xP'_\ell + \ell(\ell+1)P_\ell = 0 , \quad -1 \leq x \leq 1,
\]

with \( P_1(1) = 1 \). Rodrigues’ formula

\[
P_\ell(x) = \frac{1}{2\ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell .
\]

First few polynomials

\[
P_0(x) = 1 , \ P_1(x) = x , \ P_2(x) = \frac{3x^2 - 1}{2} , \ P_3(x) = \frac{5x^3 - 3x}{2} , \ P_4(x) = \frac{35x^4 - 30x^2 + 3}{8} .
\]

Recursion relation:

\[
\ell P_\ell(x) = (2\ell - 1) x P_{\ell-1}(x) - (\ell - 1) P_{\ell-2}(x) .
\]

Generating function:

\[
\Phi(x,h) = \frac{1}{\sqrt{1 - 2xh + h^2}} = \sum_{\ell=0}^{\infty} h^\ell P_\ell(x) .
\]

Orthogonality:

\[
\int_{-1}^{1} P_n(x)P_m(x) \, dx = \delta_{nm} \frac{2}{2n + 1} .
\]
Legendre Series:

\[ f(x) = \sum_{n=0}^{\infty} c_n P_n(x) , \quad c_n = \frac{2n + 1}{2} \int_{-1}^{1} P_n(x) f(x) dx . \]

Bessel’s equation and functions: \( x^2 y'' + xy' + (x^2 - p^2)y = 0 \) has solutions \( J_p(x) \) and \( N_p(x) \), with \( J_p(x) \propto x^p \) for small \( x \).

Orthogonality (for \( p \) an integer \( \geq 0 \))

\[ \int_{0}^{1} x J_{p}(\alpha_{p,n}x) J_{p}(\alpha_{p,m}x) dx = \delta_{nm} \frac{J_{p+1}^2(\alpha_{p,m})}{2} \]

where \( \alpha_{p,n} \) is \( n \)’th zero of \( J_p(x) \).

Gamma and Beta functions:

\[ \Gamma(p) = \int_{0}^{\infty} x^{p-1} e^{-x} dx , \quad [\text{Re}(p) > 0] ; \quad \Gamma(p + 1) = p \Gamma(p) , \quad \Gamma(1) = 1 , \quad \Gamma(1/2) = \sqrt{\pi} , \]

\[ \Gamma(p) \Gamma(1 - p) = \frac{\pi}{\sin(\pi p)} , \quad B(p, q) = \int_{0}^{1} x^{p-1}(1 - x)^{q-1} dx = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p + q)} , \quad [\text{Re}(p, q) > 0] \]

Laplacian:

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

Orthogonality of Trigonometric functions.

\[ \int_{0}^{L} \sin(\pi mx/L) \sin(\pi nx/L) dx = \delta_{mn} L/2 \quad (n, m > 0) , \]
\[ \int_{-L/2}^{L/2} \cos(\pi mx/L) \cos(\pi nx/L) dx = \delta_{mn} L/2 \quad (n, m > 0) , \]
\[ \int_{0}^{L} e^{-2i\pi m/L} e^{2i\pi n/L} dx = \delta_{mn} L \]