parametric equation problems (on webpage)

#1) \( x = \sqrt{t - 4} \) \( \Rightarrow \) \( y = t + 1 \)
\[
\frac{dx}{dt} = \frac{1}{2\sqrt{t-4}}; \quad \frac{dy}{dt} = 1
\]
\[
\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} \quad \text{ans} \quad y' = 2\sqrt{t-4}
\]
\( \text{Note: } t = y - 1, \text{ so } x^2 = t - 4 = y - 5 \)
\( \Rightarrow y = x^2 + 5 \) \( \text{ and } \)
\( y' = 2x = 2\sqrt{t-4} \)

#2) \( x = \sqrt{t} + 1 \); \( y = \sqrt[3]{t} = t^{1/3} \)
\[
\frac{dx}{dt} = \frac{1}{2\sqrt{t}}; \quad \frac{dy}{dt} = \frac{1}{3} t^{-2/3}
\]
\[
\frac{dy}{dx} = \frac{(1/3) t^{-2/3}}{(1/2) t^{-1/2}} \quad \text{ans} \quad y' = \frac{2}{3} t^{-1/6}
\]
\( \text{Note: } t = (x-1)^2 \)
\( \Rightarrow y = (x-1)^{2/3} \) \( \text{ or } \)
\( y' = \frac{2}{3} (x-1)^{-1/3} \) \( \text{ or } \)
\( y' = \frac{2}{3} [t^{1/2}]^{-1/3} = \frac{2}{3} t^{-1/6} \)

#3) \( x = 5(1-e^{-t}) \); \( y = 7(1-e^{-t}) - 3t \)
\( \dot{x} = 5e^{-t} \) \( \dot{y} = 7e^{-t} - 3 \)
\[
\frac{dy}{dx} = \frac{7e^{-t} - 3}{5e^{-t}} \quad \text{ans} \quad y' = \frac{7}{5} - \frac{3}{5} e^{t}
\]

#4) \( x = \frac{1-t^2}{1+t^2}; \quad y = \frac{2t}{1+t^2} \)
\( \dot{x} = \frac{-4t}{(1+t^2)^2}; \quad \dot{y} = \frac{2(1-t^2)}{(1+t^2)^2} \)
\( \text{Q.R.,} \)
\[
\frac{dy}{dx} = -\frac{1}{2} \left( \frac{1-t^2}{t} \right) \quad \text{Note: since } 1-t^2 = x(1+t^2) \text{ and } 2t = y(1+t^2) \]
\( y' = -\frac{x}{y} \)

1/4
#2) \( z = f(x, y) = 2x \sqrt{y} - \left(3/x^2y^2\right) \)

Let: \( y \to b \), then

\[ z = (2\sqrt{b}) x - (3/b^2) \cdot \frac{1}{x}; \]

\[ \frac{dz}{dx} = 2 \sqrt{b} - (3/b^2)(-1/x^2); \]

Replace \( b \to y \), and

\[ \frac{\partial z}{\partial x} = 2 \sqrt{y} - \left(\frac{3}{x^2y}\right) \text{ ans} \]

Let: \( x \to a \), then

\[ z = (2a) \sqrt{y} - \left(\frac{3}{a^2}\right) \left(\frac{1}{y^2}\right); \]

\[ \frac{dz}{dy} = (2a) \left(\frac{1}{2 \sqrt{y}}\right) - \left(\frac{3}{a^2}\right) \left(-\frac{2}{y^3}\right) \]

Replace \( a \to x \), and

\[ \frac{\partial z}{\partial y} = \frac{x}{\sqrt{y}} + \frac{6}{x^2y^3} \text{ ans} \]

\# 6) \( z = f(x, y) = \tan(x - 2y) \);

Let: \( w = x - 2b \)

\[ z = \tan(w) \text{ c.r.} \]

\[ \frac{dz}{dx} = \frac{dz}{dw} \cdot \frac{dw}{dx} = \frac{1}{\cos^2(w)} \cdot (4) \]

\[ \frac{\partial z}{\partial x} = \frac{1}{\cos^2(x - 2y)} = \sec^2(x - 2y) \text{ ans} \]

Let: \( w = a - 2y \)

\[ z = \tan(w) \text{ c.r.} \]

\[ \frac{dz}{dy} = \frac{dz}{dw} \cdot \frac{dw}{dy} = \frac{1}{\cos^2(w)} \cdot (-2) \]

\[ \frac{\partial z}{\partial y} = \frac{-2}{\cos^2(x - 2y)} = -2 \sec^2(x - 2y) \text{ ans} \]

Note: Neu 2nd ed: \( z = \tan(x - y) \)

\[ \frac{\partial z}{\partial x} = \frac{1}{\cos^2(x - y)} = \sec^2(x - y) \]

\[ \frac{\partial z}{\partial y} = \frac{-1}{\cos^2(x - y)} = -\sec^2(x - y) \]
#10) \( z = f(x, y) = x^2 e^{-\frac{1}{2}xy} \)
Let: \( z = x^2 e^{-\frac{b}{2}y} \), then
\[
\frac{dz}{dx} = (2x)e^{-\frac{b}{2}y} + (x^2)(-\frac{b}{2}e^{-\frac{b}{2}y})
\]
\[
\frac{\partial z}{\partial x} = 2x \left( 1 - \frac{1}{4} xy \right) e^{-\frac{1}{2}xy}
\]
\[
\begin{align*}
\text{ans} & \ \\
\end{align*}
\]
Note: New 2nd ed.: \( z = x^2 e^{-xy} \)
\[
\frac{\partial z}{\partial x} = 2x \left( 1 - \frac{1}{2} xy \right) e^{-xy}
\]
\[
\frac{\partial z}{\partial y} = -x^3 e^{-xy}
\]
\[
\begin{align*}
\text{ans} & \ \\
\end{align*}
\]

#12) \( z = f(x, y) = \cos(x^2 - y^2) e^{-\frac{1}{2}y^2} \)
Let: \( w = x^2 - b^2 \)
\[ z = \cos(w) e^{-b^2} \]
\[
\frac{dz}{dx} = (e^{-b^2}) \left[ \frac{d}{dw} \cos(w) \cdot \frac{dw}{dx} \right]
\]
\[
\frac{\partial z}{\partial x} = -2x \cdot \sin(x^2 - y^2) \cdot e^{-\frac{1}{2}y^2}
\]
\[
\begin{align*}
\text{ans} & \ \\
\end{align*}
\]
Note: New 2nd ed.: \( z = \cos(x^2 + y^2) e^{-\frac{1}{2}y^2} \)
\[
\frac{\partial z}{\partial x} = -2x \cdot \sin(x^2 + y^2) \cdot e^{-\frac{1}{2}y^2}
\]
\[
\begin{align*}
\text{ans} & \ \\
\end{align*}
\]
\[
\frac{\partial z}{\partial y} = -2y \left[ \sin(x^2 + y^2) + \cos(x^2 + y^2) \right] e^{-\frac{1}{2}y^2}
\]
\[
\begin{align*}
\text{ans} & \ \\
\end{align*}
\]
#16) \( y = \sqrt{1+x^2} \)
\[ y' = \frac{x}{\sqrt{1+x^2}} \]
\[ y'' = 1 \left( \frac{1}{\sqrt{1+x^2}} \right)^3 \]
Set \( y' = 0 \), then crit. pt. is at \( x = 0 \)
Since \( y''(0) = +1 \), \( y \) has a local \text{MIN} at \( x = 0 \)
coordinates of \text{MIN} pt.: \( (0,1) \)

\text{Note}: No \text{MAX} pts; local \text{MIN} is the same
as the absolute \text{MIN} for this problem.
(for \(-\infty < x < 0 \), \( y \) decreases; for \( 0 \leq x < +\infty \), \( y \) increases)

#28) \( y = x^4 - 2x^2 \) or
\[ y = x^2(x-\sqrt{2})(x+\sqrt{2}) \]
\[ y' = 4x^3 - 4x = 4x(x-1)(x+1) \]
\[ y'' = 12x^2 - 4 = 12 (x-\frac{1}{\sqrt{3}})(x+\frac{1}{\sqrt{3}}) \]
\[ y''' = 24x \]
Set \( y' = 0 \), three crit. pts: 1) \( x = 0,2.) x = -1,3) x = +1 \)
at \( x = 0, y = 0 \); \( y''(0) = -4 \); \( y \) has a local \text{MAX}
at \( x = -1, y = -1 \); \( y''(-1) = +8 \); \( y \) has a local \text{MIN}
at \( x = +1, y = -1 \); \( y''(+1) = +8 \); \( y \) has a local \text{MIN}

The absolute \text{MIN} is at either \( x = -1 \) or \( x = +1 \)
There is no absolute \text{MAX}
(y dec. in \(-\infty < x < -1 \) and in \( 0 < x < +1 \);
\( y \) inc. in \(-1 < x < 0 \) and in \( 1 < x < +\infty \))

\text{Note}: infl. pts at \( x = \pm \frac{1}{\sqrt{3}} \)