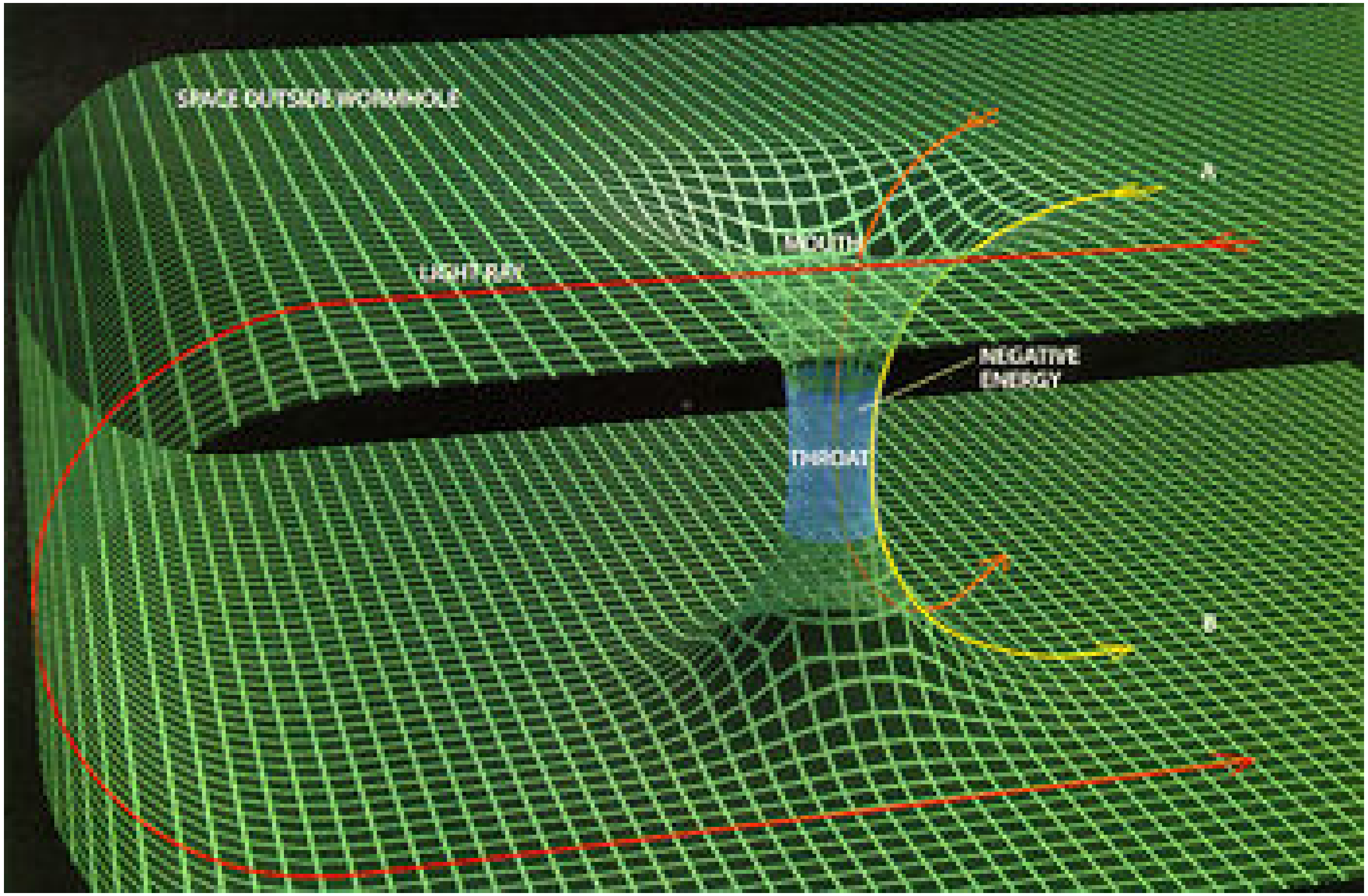

SPACE OUTSIDE WORMHOLE
A
MOUTH
LIGHT RAY
NEGATIVE
ENERGY
THROAT
B

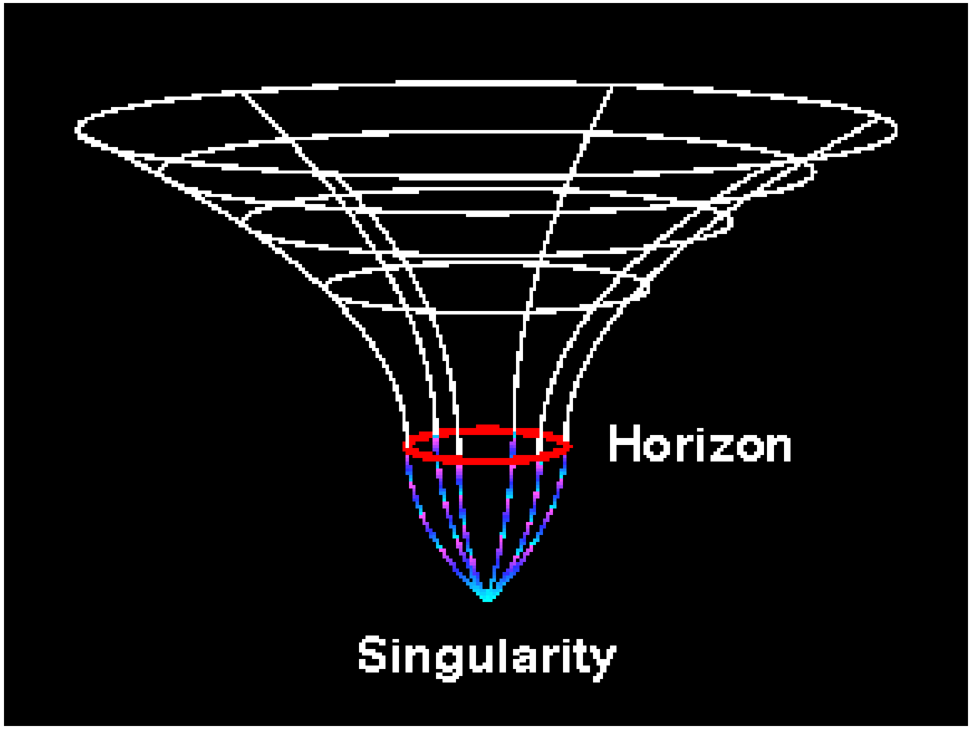
Physics 311
General Relativity

Lecture 15:
Metrics and curved space

NO HOMEWORK this week

Today's lecture plan

- Flat spacetime of Special Relativity.
- Solving Einstein Field Equation for empty space – the "vacuum solution"
- Schwarzschild metric



A look back

• In Special Relativity the spacetime is said to be “flat”, it has no “curvature”. What do we mean when we say “the spacetime is flat”, “the spacetime has no curvature”?

• We mean that the path of a free particle is a straight line, and that the square of the interval is a *linear* combination of the space and time components squared:

$$ds^2 = dt^2 - (dx^2 + dy^2 + dz^2)$$

(in the system of units where c = 1)

• This is a lot like the Euclidean geometry, which is also flat. We’ve alluded to a non-Euclidean geometry in the last lecture; we’ll soon see how it comes to be.

The Minkowski metric

• We can write the expression for the interval in the matrix form:

$$\mathbf{ds} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \times \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}$$

(so that $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$)

• This matrix – a very simple matrix indeed – defines the metric of Special Relativity, the **Minkowski metric**. It is simple yet powerful; it completely describes the spacetime of Special Relativity.

Einstein Field Equation: another dissection

- Generally speaking, Einstein field equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

is a set of 10 (16 components in each tensor, minus 6 due to symmetry, and if you're smart enough, minus 4 more) *coupled elliptic-hyperbolic nonlinear partial differential equations for the metric components*.

- Just so that we are clear on definitions:

"coupled" – each differential equation contains multiple terms; equations cannot be solved individually.

"elliptic-hyperbolic" – determinants of sub-matrices of the system of equations matrix are either positive or negative; never zero.

"nonlinear" – dependent on nonlinear function of metric components

"partial differential equation" – an equation containing partial derivatives of functions, for example $\partial^2 f(x,y,z)/\partial x \partial y$

"metric components" – components of the metric tensor $g_{\mu\nu}$

Einstein Field Equation expanded

- From "simple" form: $\mathbf{G_{\mu\nu} = 8\pi T_{\mu\nu}}$

$R_{\mu\nu}$

...to rather more complex, expanded form...

$$[\partial_\eta(\Gamma_{\mu}{}^{\eta}{}_{\nu}) - \partial_\nu(\Gamma_{\eta}{}^{\eta}{}_{\mu}) + \Gamma_{\eta}{}^{\eta}{}_{\lambda}\,\Gamma_{\mu}{}^{\lambda}{}_{\nu} - \Gamma_{\mu}{}^{\lambda}{}_{\eta}\,\Gamma_{\nu}{}^{\eta}{}_{\lambda}] -$$

R

$$-\ ½\, g^{\alpha\beta}\,[\partial_\eta(\Gamma_{\alpha}{}^{\eta}{}_{\beta}) - \partial_\beta(\Gamma_{\eta}{}^{\eta}{}_{\alpha}) + \Gamma_{\eta}{}^{\eta}{}_{\lambda}\,\Gamma_{\alpha}{}^{\lambda}{}_{\beta} - \Gamma_{\alpha}{}^{\lambda}{}_{\eta}\,\Gamma_{\beta}{}^{\eta}{}_{\lambda}]\, g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Here, $\Gamma_{\alpha}{}^{\gamma}{}_{\beta} = ½\, g^{\delta\gamma}[\,\partial_\beta(g_{\alpha\delta}) + \partial_\alpha(g_{\beta\delta}) - \partial_\delta(g_{\alpha\beta})]$ are *Christoffel symbols of 2nd kind* – tensor-like objects derived from Riemann metric g;

$\partial_\alpha = (\partial/\partial x^\alpha)$ denotes partial derivative with respect to variable x^α;

and $g_{\alpha\beta}$ is the metric tensor – roughly speaking, the function that tells us how to compute distances between points in a given space:

$$ds^2 = \sum g_{\alpha\beta}\, dx_\alpha dx_\beta$$

Back to Minkowski metric

• The matrix in the expression for the interval is nothing more, nor less, than the Minkowski metric tensor $g_{\alpha\beta}$

$$\mathbf{ds} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \times \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}$$

• So we actually know one metric tensor already – it's not too scary at all!

… but that’s a very special case…

... a slightly more general case...

... looks a lot different...

If you sit down and write down the Ricci tensor for a general case of a 2-dimensional space with axial symmetry, you would get something like this:

$$
\begin{aligned}
R_{\theta\eta} = & -\frac{2\,a^2\,\frac{\partial\psi}{\partial\theta}\,\cot\theta}{\delta\,\psi} + \frac{2\,a\,c\,\frac{\partial\psi}{\partial\eta}\,\cot\theta}{\delta\,\psi} + \frac{a\,\frac{\partial c}{\partial\eta}\,\cot\theta}{\delta} - \frac{\frac{\partial a}{\partial\eta}\,c\,\cot\theta}{2\,\delta} - \frac{a\,\frac{\partial a}{\partial\theta}\,\cot\theta}{2\,\delta} - \frac{2\,a^2\,\frac{\partial^2\psi}{\partial\theta^2}}{\delta\,\psi} \\
& - \frac{2\,a^2\,(\frac{\partial\psi}{\partial\theta})^2}{\delta\,\psi^2} + \frac{4\,a\,c\,\frac{\partial\psi}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi^2} - \frac{a^2\,\frac{\partial d}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,d\,\psi} + \frac{a\,c\,\frac{\partial d}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,d\,\psi} + \frac{2\,a\,\frac{\partial c}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi} - \frac{\frac{\partial a}{\partial\eta}\,c\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi} \\
& - \frac{3\,a\,\frac{\partial a}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta\,\psi} - \frac{2\,a^2\,c\,\frac{\partial c}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} + \frac{2\,a^2\,b\,\frac{\partial c}{\partial\eta}\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} - \frac{a^2\,\frac{\partial b}{\partial\eta}\,c\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} - \frac{a\,\frac{\partial a}{\partial\eta}\,b\,c\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} + \frac{a^3\,\frac{\partial b}{\partial\theta}\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} \\
& + \frac{a^2\,\frac{\partial a}{\partial\theta}\,b\,\frac{\partial\psi}{\partial\theta}}{\delta^2\,\psi} - \frac{2\,a\,b\,\frac{\partial^2\psi}{\partial\eta^2}}{\delta\,\psi} - \frac{2\,\frac{\partial^2\psi}{\partial\eta^2}}{\psi} + \frac{4\,a\,c\,\frac{\partial^2\psi}{\partial\eta\partial\theta}}{\delta\,\psi} - \frac{2\,a\,b\,(\frac{\partial\psi}{\partial\eta})^2}{\delta\,\psi^2} + \frac{6\,(\frac{\partial\psi}{\partial\eta})^2}{\psi^2} \\
& + \frac{a\,c\,\frac{\partial d}{\partial\theta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,d\,\psi} - \frac{a\,b\,\frac{\partial d}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,d\,\psi} - \frac{2\,c\,\frac{\partial c}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} + \frac{\frac{\partial a}{\partial\theta}\,c\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} - \frac{2\,a\,\frac{\partial b}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} + \frac{\frac{\partial a}{\partial\eta}\,b\,\frac{\partial\psi}{\partial\eta}}{\delta\,\psi} \\
& + \frac{2\,a^2\,b\,\frac{\partial c}{\partial\theta}\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} - \frac{2\,a\,b\,c\,\frac{\partial c}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} - \frac{a^2\,\frac{\partial b}{\partial\theta}\,c\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} - \frac{a\,\frac{\partial a}{\partial\theta}\,b\,c\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} + \frac{a^2\,b\,\frac{\partial b}{\partial\eta}\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} + \frac{a\,\frac{\partial a}{\partial\eta}\,b^2\,\frac{\partial\psi}{\partial\eta}}{\delta^2\,\psi} \\
& + \frac{a\,\frac{\partial c}{\partial\eta}\,\frac{\partial d}{\partial\theta}}{2\,\delta\,d} - \frac{\frac{\partial a}{\partial\eta}\,c\,\frac{\partial d}{\partial\theta}}{4\,\delta\,d} - \frac{a\,\frac{\partial a}{\partial\theta}\,\frac{\partial d}{\partial\theta}}{4\,\delta\,d} - \frac{\frac{\partial^2 d}{\partial\eta^2}}{2\,d} + \frac{(\frac{\partial d}{\partial\eta})^2}{4\,d^2} - \frac{c\,\frac{\partial c}{\partial\eta}\,\frac{\partial d}{\partial\eta}}{2\,\delta\,d} \\
& + \frac{\frac{\partial a}{\partial\theta}\,c\,\frac{\partial d}{\partial\eta}}{4\,\delta\,d} + \frac{\frac{\partial a}{\partial\eta}\,b\,\frac{\partial d}{\partial\eta}}{4\,\delta\,d} + \frac{a\,\frac{\partial^2 c}{\partial\eta\partial\theta}}{\delta} - \frac{a\,\frac{\partial^2 b}{\partial\eta^2}}{2\,\delta} - \frac{a\,\frac{\partial^2 a}{\partial\theta^2}}{2\,\delta} + \frac{a\,c\,\frac{\partial c}{\partial\eta}\,\frac{\partial c}{\partial\theta}}{\delta^2}
\end{aligned}
$$

$$-\frac{a\,\frac{\partial a}{\partial \theta}\,c\,\frac{\partial c}{\partial \theta}}{2\,\delta^2}-\frac{a\,\frac{\partial a}{\partial \eta}\,b\,\frac{\partial c}{\partial \theta}}{2\,\delta^2}-\frac{a\,\frac{\partial b}{\partial \eta}\,c\,\frac{\partial c}{\partial \eta}}{2\,\delta^2}-\frac{a^2\,\frac{\partial b}{\partial \theta}\,\frac{\partial c}{\partial \eta}}{2\,\delta^2}+\frac{a\,\frac{\partial a}{\partial \eta}\,\frac{\partial b}{\partial \theta}\,c}{4\,\delta^2}-\frac{a\,\frac{\partial a}{\partial \theta}\,\frac{\partial b}{\partial \eta}\,c}{4\,\delta^2}$$

$$+\frac{a^2\,\frac{\partial a}{\partial \theta}\,\frac{\partial b}{\partial \theta}}{4\,\delta^2}+\frac{a^2\,(\frac{\partial b}{\partial \eta})^2}{4\,\delta^2}+\frac{a\,\frac{\partial a}{\partial \eta}\,b\,\frac{\partial b}{\partial \eta}}{4\,\delta^2}+\frac{a\,(\frac{\partial a}{\partial \theta})^2\,b}{4\,\delta^2}$$

$$R_{\eta\theta}=-\frac{2\,a\,c\,\frac{\partial \psi}{\partial \theta}\,\cot\theta}{\delta\,\psi}+\frac{2\,a\,b\,\frac{\partial \psi}{\partial \eta}\,\cot\theta}{\delta\,\psi}-\frac{2\,\frac{\partial \psi}{\partial \eta}\,\cot\theta}{\psi}-\frac{\frac{\partial d}{\partial \eta}\,\cot\theta}{2\,d}-\frac{\frac{\partial a}{\partial \theta}\,c\,\cot\theta}{2\,\delta}+\frac{a\,\frac{\partial b}{\partial \eta}\,\cot\theta}{2\,\delta}$$

$$-\frac{2\,a\,c\,\frac{\partial^2 \psi}{\partial \theta^2}}{\delta\,\psi}-\frac{2\,a\,c\,(\frac{\partial \psi}{\partial \theta})^2}{\delta\,\psi^2}+\frac{4\,a\,b\,\frac{\partial \psi}{\partial \eta}\,\frac{\partial \psi}{\partial \theta}}{\delta\,\psi^2}+\frac{2\,\frac{\partial \psi}{\partial \eta}\,\frac{\partial \psi}{\partial \theta}}{\psi^2}-\frac{a\,c\,\frac{\partial d}{\partial \theta}\,\frac{\partial \psi}{\partial \theta}}{\delta\,d\,\psi}+\frac{a\,b\,\frac{\partial d}{\partial \eta}\,\frac{\partial \psi}{\partial \theta}}{\delta\,d\,\psi}$$

$$-\frac{\frac{\partial d}{\partial \eta}\,\frac{\partial \psi}{\partial \theta}}{d\,\psi}+\frac{2\,a\,\frac{\partial c}{\partial \theta}\,\frac{\partial \psi}{\partial \theta}}{\delta\,\psi}-\frac{3\,\frac{\partial a}{\partial \theta}\,c\,\frac{\partial \psi}{\partial \theta}}{\delta\,\psi}+\frac{2\,a\,\frac{\partial b}{\partial \eta}\,\frac{\partial \psi}{\partial \theta}}{\delta\,\psi}+\frac{\frac{\partial a}{\partial \eta}\,b\,\frac{\partial \psi}{\partial \theta}}{\delta\,\psi}-\frac{2\,a^2\,b\,\frac{\partial c}{\partial \theta}\,\frac{\partial \psi}{\partial \theta}}{\delta^2\,\psi}$$

$$+\frac{2\,a\,b\,c\,\frac{\partial c}{\partial \eta}\,\frac{\partial \psi}{\partial \theta}}{\delta^2\,\psi}+\frac{a^2\,\frac{\partial b}{\partial \theta}\,c\,\frac{\partial \psi}{\partial \theta}}{\delta^2\,\psi}+\frac{a\,\frac{\partial a}{\partial \theta}\,b\,c\,\frac{\partial \psi}{\partial \theta}}{\delta^2\,\psi}-\frac{a^2\,b\,\frac{\partial b}{\partial \eta}\,\frac{\partial \psi}{\partial \theta}}{\delta^2\,\psi}-\frac{a\,\frac{\partial a}{\partial \eta}\,b^2\,\frac{\partial \psi}{\partial \theta}}{\delta^2\,\psi}-\frac{2\,b\,c\,\frac{\partial^2 \psi}{\partial \eta^2}}{\delta\,\psi}$$

$$+\frac{4\,a\,b\,\frac{\partial^2 \psi}{\partial \eta\,\partial \theta}}{\delta\,\psi}-\frac{6\,\frac{\partial^2 \psi}{\partial \eta\,\partial \theta}}{\psi}-\frac{2\,b\,c\,(\frac{\partial \psi}{\partial \eta})^2}{\delta\,\psi^2}+\frac{a\,b\,\frac{\partial d}{\partial \theta}\,\frac{\partial \psi}{\partial \eta}}{\delta\,d\,\psi}-\frac{\frac{\partial d}{\partial \theta}\,\frac{\partial \psi}{\partial \eta}}{d\,\psi}-\frac{b\,c\,\frac{\partial d}{\partial \eta}\,\frac{\partial \psi}{\partial \eta}}{\delta\,d\,\psi}$$

$$+\frac{2\,b\,\frac{\partial c}{\partial \eta}\,\frac{\partial \psi}{\partial \eta}}{\delta\,\psi}-\frac{3\,\frac{\partial b}{\partial \eta}\,c\,\frac{\partial \psi}{\partial \eta}}{\delta\,\psi}+\frac{a\,\frac{\partial b}{\partial \theta}\,\frac{\partial \psi}{\partial \eta}}{\delta\,\psi}+\frac{2\,\frac{\partial a}{\partial \theta}\,b\,\frac{\partial \psi}{\partial \eta}}{\delta\,\psi}+\frac{2\,a\,b\,c\,\frac{\partial c}{\partial \theta}\,\frac{\partial \psi}{\partial \eta}}{\delta^2\,\psi}-\frac{2\,a\,b^2\,\frac{\partial c}{\partial \eta}\,\frac{\partial \psi}{\partial \eta}}{\delta^2\,\psi}$$

... and just a little bit more.

$$+\frac{a\,b\,\frac{\partial b}{\partial \eta}\,c\,\frac{\partial \psi}{\partial \eta}}{\delta^2\,\psi}+\frac{\frac{\partial a}{\partial \eta}\,b^2\,c\,\frac{\partial \psi}{\partial \eta}}{\delta^2\,\psi}-\frac{a^2\,b\,\frac{\partial b}{\partial \theta}\,\frac{\partial \psi}{\partial \eta}}{\delta^2\,\psi}-\frac{a\,\frac{\partial a}{\partial \theta}\,b^2\,\frac{\partial \psi}{\partial \eta}}{\delta^2\,\psi}+\frac{\frac{\partial d}{\partial \eta}\,\frac{\partial d}{\partial \theta}}{4\,d^2}-\frac{\frac{\partial a}{\partial \theta}\,c\,\frac{\partial d}{\partial \theta}}{4\,\delta\,d}$$

$$+\frac{a\,\frac{\partial b}{\partial \eta}\,\frac{\partial d}{\partial \theta}}{4\,\delta\,d}-\frac{\frac{\partial^2 d}{\partial \eta\,\partial \theta}}{2\,d}-\frac{\frac{\partial b}{\partial \eta}\,c\,\frac{\partial d}{\partial \eta}}{4\,\delta\,d}+\frac{\frac{\partial a}{\partial \theta}\,b\,\frac{\partial d}{\partial \eta}}{4\,\delta\,d}-\frac{\frac{\partial c}{\partial \eta}\,\frac{\partial c}{\partial \theta}}{\delta}+\frac{\frac{\partial a}{\partial \theta}\,\frac{\partial c}{\partial \theta}}{2\,\delta}$$

$$+\frac{c\,\frac{\partial^2 c}{\partial \eta\,\partial \theta}}{\delta}+\frac{\frac{\partial b}{\partial \eta}\,\frac{\partial c}{\partial \eta}}{2\,\delta}-\frac{\frac{\partial^2 b}{\partial \eta^2}\,c}{2\,\delta}-\frac{\frac{\partial^2 a}{\partial \theta^2}\,c}{2\,\delta}-\frac{\frac{\partial a}{\partial \eta}\,\frac{\partial b}{\partial \theta}}{4\,\delta}+\frac{\frac{\partial a}{\partial \theta}\,\frac{\partial b}{\partial \eta}}{4\,\delta}$$

$$+\frac{a\,b\,\frac{\partial c}{\partial \eta}\,\frac{\partial c}{\partial \theta}}{\delta^2}-\frac{\frac{\partial a}{\partial \eta}\,b\,c\,\frac{\partial c}{\partial \theta}}{2\,\delta^2}-\frac{a\,\frac{\partial a}{\partial \theta}\,b\,\frac{\partial c}{\partial \theta}}{2\,\delta^2}-\frac{a\,\frac{\partial b}{\partial \theta}\,c\,\frac{\partial c}{\partial \eta}}{2\,\delta^2}-\frac{a\,b\,\frac{\partial b}{\partial \eta}\,\frac{\partial c}{\partial \eta}}{2\,\delta^2}+\frac{a\,\frac{\partial a}{\partial \theta}\,\frac{\partial b}{\partial \theta}\,c}{4\,\delta^2}$$

$$+\frac{a\,(\frac{\partial b}{\partial \eta})^2\,c}{4\,\delta^2}+\frac{\frac{\partial a}{\partial \eta}\,b\,\frac{\partial b}{\partial \eta}\,c}{4\,\delta^2}+\frac{(\frac{\partial a}{\partial \theta})^2\,b\,c}{4\,\delta^2}+\frac{a\,\frac{\partial a}{\partial \eta}\,b\,\frac{\partial b}{\partial \theta}}{4\,\delta^2}-\frac{a\,\frac{\partial a}{\partial \theta}\,b\,\frac{\partial b}{\partial \eta}}{4\,\delta^2}$$

$$R_{\phi\phi}=-\frac{2\,a\,b\,\frac{\partial \psi}{\partial \theta}\,\cot\theta}{\delta\,\psi}+\frac{2\,b\,c\,\frac{\partial \psi}{\partial \eta}\,\cot\theta}{\delta\,\psi}-\frac{\frac{\partial d}{\partial \theta}\,\cot\theta}{d}-\frac{c\,\frac{\partial c}{\partial \theta}\,\cot\theta}{\delta}+\frac{\frac{\partial b}{\partial \eta}\,c\,\cot\theta}{2\,\delta}+\frac{a\,\frac{\partial b}{\partial \theta}\,\cot\theta}{2\,\delta}$$

$$-\frac{2\,a\,b\,\frac{\partial^2 \psi}{\partial \theta^2}}{\delta\,\psi}-\frac{2\,\frac{\partial^2 \psi}{\partial \theta^2}}{\psi}-\frac{2\,a\,b\,(\frac{\partial \psi}{\partial \theta})^2}{\delta\,\psi^2}+\frac{6\,(\frac{\partial \psi}{\partial \theta})^2}{\psi^2}+\frac{4\,b\,c\,\frac{\partial \psi}{\partial \eta}\,\frac{\partial \psi}{\partial \theta}}{\delta\,\psi^2}-\frac{a\,b\,\frac{\partial d}{\partial \theta}\,\frac{\partial \psi}{\partial \theta}}{\delta\,d\,\psi}$$

This is a general expression for Ricci tensor R_{mn} in only two dimensions, with axial symmetry. (From Larry Smarr, Univ. of Illinois)

Just try to imagine all of three dimensions of space plus one of time!

Special case: vacuum

• I haven't said anything about the energy-stress tensor $T_{\mu\nu}$ yet. Well, here's an example of this tensor:

$$T_{\mu\nu} = 0$$

• This special case is called "vacuum", and corresponding solutions for the metric $g_{\mu\nu}$ are called "vacuum solutions". In this case, we have $R = 0$.

• Wait – what solutions? If we set $T_{\mu\nu}$ to zero, wouldn't our metric be just zero as well?

• Not really! The field equation now has form:

$$\mathbf{R}_{\mu\nu} = \mathbf{0}$$

But the left-hand side is a **complicated mess** of derivatives of the metric. There can be many solutions for this "vacuum" equation, including several *exact analytic* solution. These different solutions arise from different symmetries we impose on the metric.

Minkowski metric (one last time)

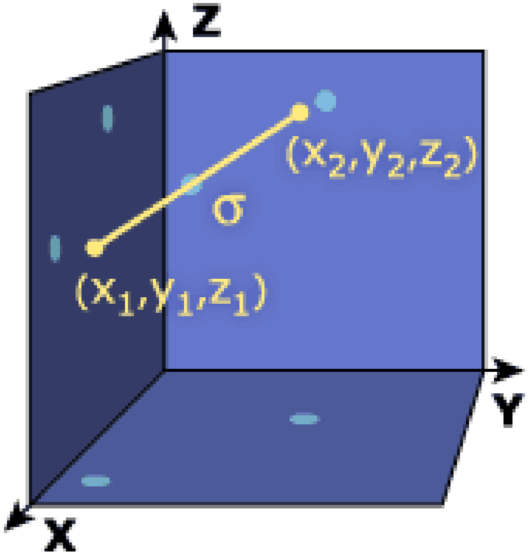
• Minkowski metric is one of the *vacuum solutions* for a space that has perfect symmetry – a space that is:

- uniform, so that $g(t,x,y,z) = g(t,x+\Delta x,y,z)$ (also true for y and z)

- isotropic, so that $g(t,x,y,z) = g(t,-x,y,z)$ (also true for y and z)

and a time that is:

- uniform, so that $g(t,x,y,z) = g(t+\Delta t,x,y,z)$



Schwarzschild Vacuum Solution

• Another important metric, first to be explicitly solved only weeks after Einstein published his General Relativity paper is 1915, is called ***Schwarzschild metric***, named after the man who solved it.

• This solution assumes *spherical symmetry* of space, as *around* an isolated star.

• How is this "vacuum" if there is a star?! There's mass, thus there is energy, and there must be stress somehow, so tensor $T_{\mu\nu}$ must be nonzero!

• The keyword is "around" – the solution is for the metric of *empty space* (also known as "vacuum") surrounding a spherically-symmetric massive object.

Karl Schwarzschild

Derivation of Schwarzschild solution. 1. Assumptions and notation

- We start by defining our assumptions and notation.

  1. The coordinates are (t, r, θ, ϕ) – time + spherical coordinate system. We call these coordinates x_μ, with $\mu = 1...4$.

  2. Spherical symmetry: metric components are unchanged under $r \rightarrow -r$, $\theta \rightarrow -\theta$ and $\phi \rightarrow -\phi$.

  3. Spacetime is static, i.e. all metric components are independent of time: $(\partial g_{\mu\nu}/\partial t) = 0$; this also means that spacetime is invariant under time reversal.

  4. We are looking for vacuum solution $T_{\mu\nu} = 0$, with $R = 0$.

- What we need to solve then is:

$$R_{\mu\nu} = 0$$

Derivation of Schwarzschild solution. 2. Diagonalizing

- The requirements that metric be time-independent, and symmetric with respect to rotations, allow us to diagonalize the matrix:

  1. Time-reversal symmetry: $(t, r, \theta, \phi) \rightarrow (-t, r, \theta, \phi)$ must conserve components of g. The components of the 1st column of the metric, $g_{\mu 1}$ $(\mu \neq 1)$, transforms under time reversal as: $g_{\mu 1} \rightarrow - g_{\mu 1}$

     Since we demand that $g_{\mu 1} = g_{\mu 1}$, then $g_{\mu 1} = 0$ for $(\mu \neq 1)$.

  2. Same reasoning for r, θ and ϕ – symmetries leads to all other non-diagonal (i.e. $\mu \neq \nu$) metric components to vanish.

- Thus, the sought metric has the form:

$$ds^2 = g_{11}dt^2 + g_{22}dr^2 + g_{33}d\theta^2 + g_{44}d\phi^2$$

Derivation of Schwarzschild solution. 3. Simplifying

• On a sphere of constant radius, and at constant time, the only spherically-symmetric combination of $d\theta^2$ and $d\phi^2$ is $C(r)(d\theta^2 + \sin^2\theta d\phi^2)$, where $C(r)$ is (a yet unknown) function of radius coordinate only. This expression above is simply the element of a spherical surface.

• For constant t, θ and ϕ (i.e. on the radial line) metric should only depend on the radius coordinate r – again, to conserve the spherical symmetry. That means that the metric components for time and radius, g_{11} and g_{22}, must be functions of r only.

• This simplifies the metric even further, to:

$$ds^2 = A(r)dt^2 + B(r)dr^2 + C(r)(d\theta^2 + \sin^2\theta d\phi^2)$$

Derivation of Schwarzschild solution. 4. Solving for components

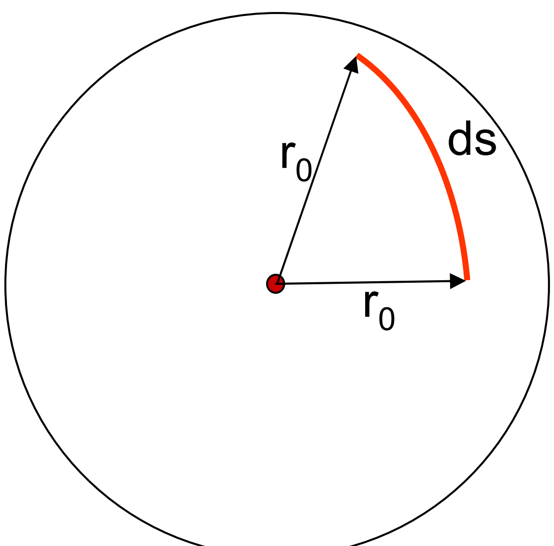
• First, we find the function $C(r)$ by noticing that at a surface of constant radius r_0 and at constant time, the separation can be written as:

$$ds^2 = r_0^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

• Since this must hold true for all radial surfaces, i.e. for any r, the unknown function $C(r)$ is simply r^2, and the θ and ϕ – components of the metric are:

$$g_{33} = r^2$$

$$g_{44} = r^2 \sin^2\theta$$



Derivation of Schwarzschild solution. 4.1. Solving for components

- Functions $A(r)$ and $B(r)$ can be found by solving the Einstein field equation (what a surprise!). Only 4 equations remain non-trivial:

$$4\partial_r AB - 2r\partial_r^2 BAB + r\partial_r A\partial_r BB + r\partial_r B^2 A = 0$$

$$r\partial_r AB + 2A^2B - 2AB - r\partial_r BA = 0$$

$$-2r\partial_r^2 BAB + r\partial_r A\partial_r BB + r\partial_r B^2 A - 4\partial_r BA = 0$$

$$(-2r\partial_r^2 BAB + r\partial_r A\partial_r BB + r\partial_r B^2 A - 4\partial_r BAB)\sin^2\theta = 0$$

- Subtracting equations 1 and 3 leads to:

$$\partial_r AB + \partial_r BA = 0 \;\Rightarrow\; A(r)B(r) = K$$ (a non-zero, real constant)

- Substituting into equation 2 we get:

$$r\partial_r A - A(1 - A) = 0 \;\Rightarrow\; A(r) = K[1 + 1/(Sr)] = g_{11}$$

$$B(r) = [1 + 1/(Sr)]^{-1} = g_{22}$$

Derivation of Schwarzschild solution. 5. Arriving at solution

- Finally, we find the coefficients K and S in the *weak-field approximation* – i.e. far away from the gravitational source. At $r \to \infty$ the spacetime must approach Minkowski spacetime, thus:

$$g_{11} = K[1 + 1/(Sr)] \to K \Rightarrow K = c^2 = 1$$

- Gravity must converge to Newtonian in the weak field. This lets us find the numerical value of the constant S:

$$S = -c^2/(2Gm) = -1/(2m)$$

where m is the mass of the central body, and G is the gravitational constant.

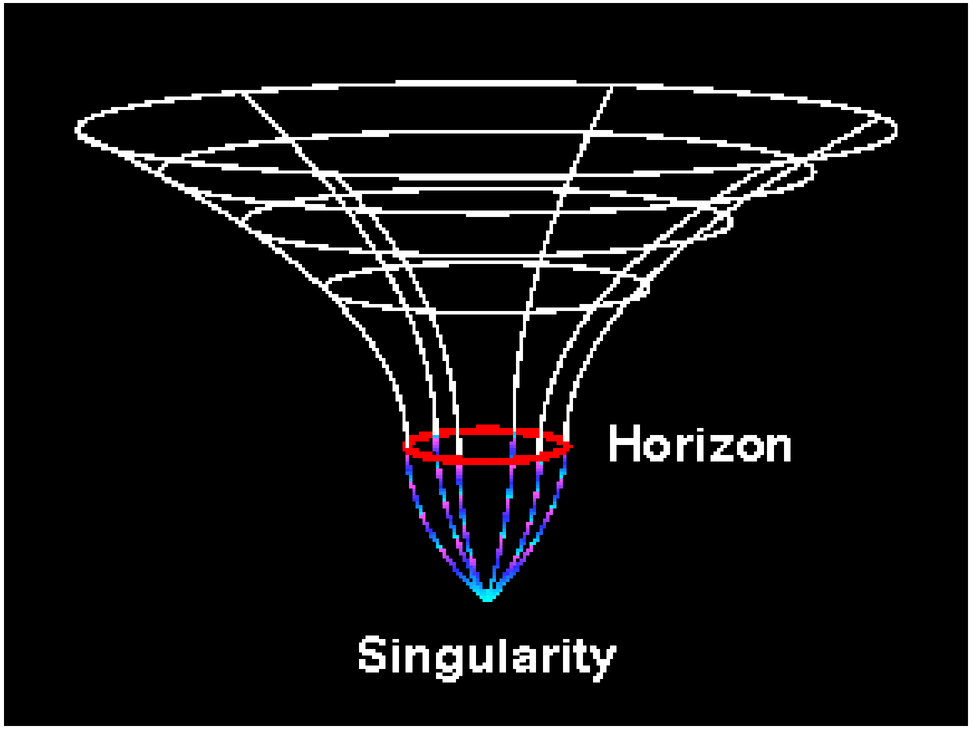
- The full Schwarzschild metric is:

$$ds^2 = [1-(2m/r)]dt^2 - (1-(2m/r))^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$$

Schwarzschild spacetime

• Schwarzschild spacetime has curvature that decreases with distance from the center. At infinity, Schwarzschild spacetime is identical to the flat Minkowski spacetime.

• In the center of Schwarzschild metric, *singularity* is possible, leading to formation of a Schwarzschild (non-rotating) *black hole*.



Recap:

• Einstein field equations can be explicitly solved for certain types of stress-energy tensor. These solutions are called *spacetime metrics*.

• Special case of stress-energy tensor – the vacuum – leads to Minkowski and Schwarzschild spacetime (among many others).

• Schwarzschild metric is fairly simple. We will mostly see its 3-dimentional (one time plus two space) case:

$$ds^2 = [1 - (2m/r)]dt^2 - [1 - (2m/r)]^{-1}dr^2 - r^2 d\phi^2$$