3-d model by Prof. H. Bülthoff - Max Planck Institute for Biological Cybernetics, Tübingen.
Animation by Ute Kraus (www.spacetimetravel.org).

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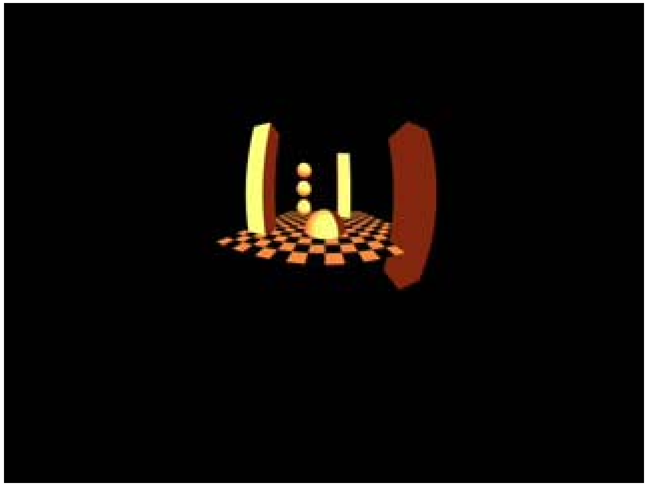
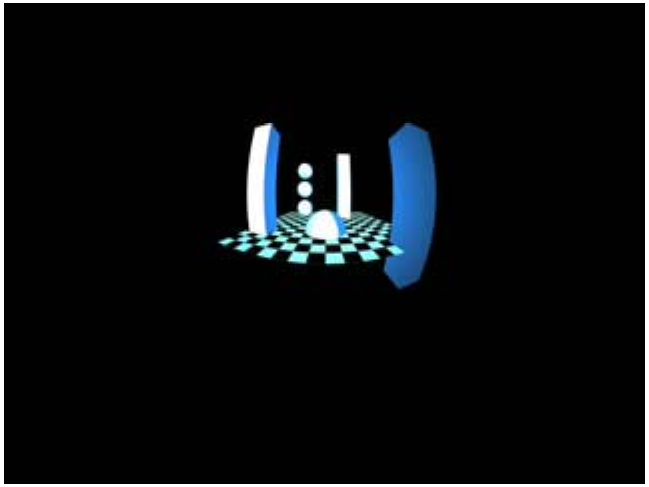




Animations by Daniel Weiskopf

4-vectors in general

• 4-vectors defined as any set of 4 quantities which transform under Lorentz transformations as does the interval. Such transformation is usually defined in the form of a matrix:

$$M = \begin{pmatrix} \gamma & -\gamma \boldsymbol{v} & 0 & 0 \\ -\gamma \boldsymbol{v} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

• The transformation for the 4-velocity is then simply:

$U = MU'$, or for its components:

$$U^0 = \gamma(U^0)' - \gamma \boldsymbol{v}(U^1)'$$
$$U^1 = -\gamma \boldsymbol{v}(U^0)' + \gamma(U^1)'$$
$$U^2 = (U^2)'$$
$$U^3 = (U^3)'$$

• Notice that the “orthogonal” components of the 4-velocity ***do not change***!

Physics 311

Special Relativity

Lecture 7:

Twin Paradox

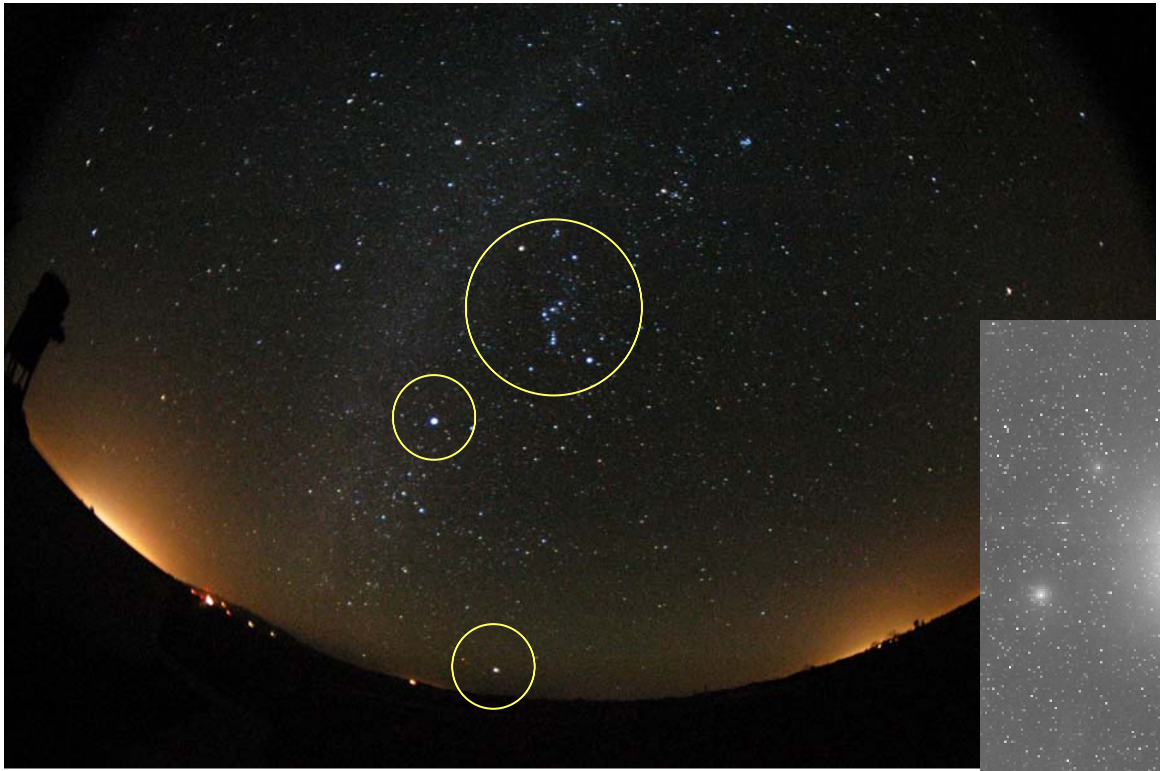
Plan of the lecture

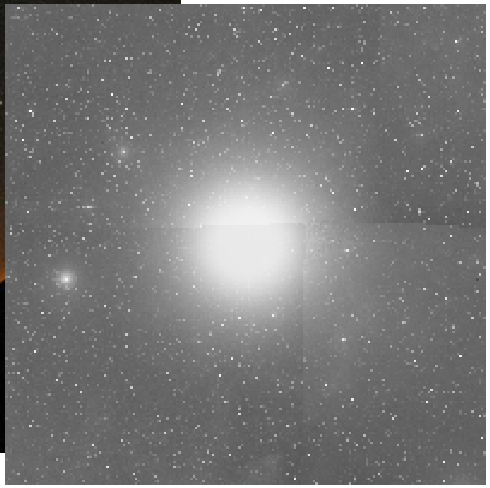
- Trip to Canopus – a star "99" light year away – on an antimatter-powered rocket ship.
- The round-trip will take at least 198 years – there's no way the crew would survive! Or is there?
- Faster than light?..
- Anywhere in the Universe in under 5 seconds!
- The Twin Paradox.
- Which twin travels? The spacetime metric.
- "Doppler shift" explanation for the Twin Paradox... Not quite satisfactory.
- Lorentz length contraction explanation... leading to a new paradox!

Canopus

• Canopus, or Alpha Carinae (Keel), is a star, approximately 313 light years from the Sun (and Earth, for that matter). The star is classified as F0Ib – a bright supergiant. It is 20,000 times brighter than our Sun.

• Taylor and Wheeler claim that their Canopus is mere 99 light years away – could they possibly mean some other mysterious Canopus?.. To avoid confusion, we'll use their number.

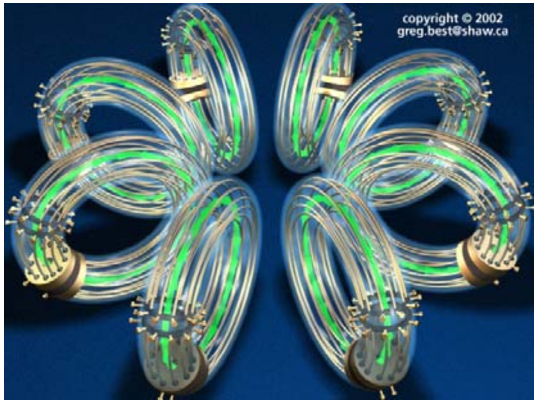




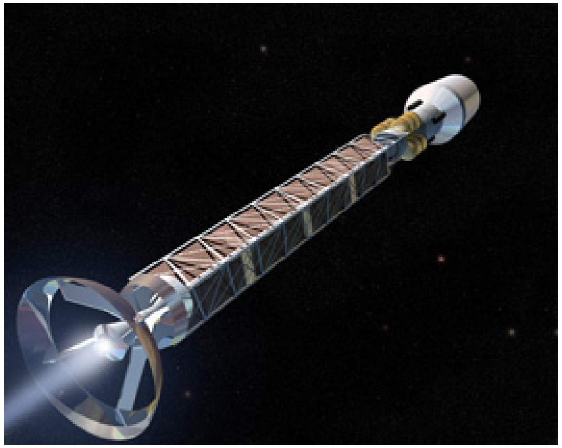
Trip to Canopus

• The "Space Agency" has decided to send an expedition to Canopus. A shiny new antimatter-propelled photon spaceship is loaded with all necessary equipment and life support for the long trip...

Photon propulsion prototype



The antimatter core



The finished product

• But how long? Surely, the spaceship cannot fly faster than light, so the trip will certainly take longer than 99 years, say, 100 years one-way, or 200 year for the round-trip.

How long???

- Will the crew survive?... 200 years is a long time!

- Of course they will! Just travel fast enough, as close to the speed of light as you can get, and time dilation will make the trip seem short for the crew, as well as for the on-board clock.

Go anywhere in any (proper) time – just go fast enough

• Going at 80% the speed of light, takes us to Canopus (99 light years away) in

$$99/0.8 = 123 \text{ years and 9 months of Earth (Lab) time,}$$

but in only

$$123.75/\gamma = 123.75(1 - 0.8^2)^{1/2} = 74 \text{ years and 3 months of proper time,}$$

thanks to the time stretch factor of 1.66666666...

• What if we go faster? The time stretch factor will get bigger, and even though the Earth frame travel time will never be shorter than 99 years, the proper time can be as short as we want!

• In the limit of the spaceship velocity ***v*** approaching the speed of light, i.e. ***v*** = 1 – ε , where ε is a small number, the proper time of the travel is:

$$\tau = [\underbrace{99/(1 - \varepsilon)}_{\text{Earth time}}][\underbrace{1 - (1 - \varepsilon)^2}_{\text{Time stretch}}]^{1/2} \approx 99(1 + \varepsilon)(1 - (1 - 2\varepsilon))^{1/2} \approx 99\sqrt{2\varepsilon}$$

How fast?

• Let's assume that our spaceship have left the Earth on the 4th of July, 2000. The ship traveled for 6 years of on-board (proper) time and reached a remote outpost – the Lookout Station 8. Number "8" stands for "8 light years from Earth".

• As we pass by the Station 8, we notice that their clock reads "07/04/2010" – this is the time in the Earth frame, the Lookout Station 8 is not moving with respect to Earth, and it's clock is properly synchronized with the Earth's.

• So, lets' see... In 6 years of proper time we've traveled 8 light years?! Our speed is (8 light years)/(6 years) = 4/3 speed of light!..

• No. Nice try, but no. Our speed in our frame – the Rocket frame – is ZERO. This is our REST frame.

• Our speed as measured in the Earth (Lab) frame is

(8 light years)/(10 years) = 0.8c – no problem!

The flight plan

- We will assume that the Rocket travels at 99/101 speed of light, or about 0.98c.
- After preliminary acceleration to 0.98c, the Rocket zooms by the Earth (this will be our Event 1). This is when both the Earth and the Rocket clocks are set to zero.
- The Rocket continues at 0.98c all the way to Canopus 99 light years away. As it passes by Canopus, Event 2 is recorded.
- The Rocket loops around Canopus without changing the speed and goes back to Earth. As we fly by Earth again, still at 0.98c, we record Event 3.
- Then the Rocket slows down and quietly lands on Earth.

The Twins

- The famous Twin Paradox: One of two twins boards a spaceship and travels to a faraway star and back at near speed of light. Due to the time dilation, the twin on the Rocket ages less than the twin on Earth. But what if we take the Rocket frame to be at rest instead, with Earth moving away and back at near speed of light??? The Earth twin should age less in this case. A contradiction?
- To solve this Paradox, we first need the Twins... Or clones?...

Which twin travels?

• The Paradox main assumption is that, according to Special Relativity, all inertial frames are equal, and thus either of the twins could have been traveling. What is wrong with this assumption?

• We need to look at the path taken by the Rocket and the Earth in ***spacetime***. At the point where the Rocket turns, the spacetime path is curved. In Lorentz spacetime geometry, thanks to its special metric, the proper time difference between two events is the *greatest* for the frame that goes along the *straight line*. All curved paths will have *shorter* proper time difference.

• The rocket frame accelerates as it turns around. This distinguishes it from the Earth free-float frame. The spacetime path of the Rocket frame is *curved*.

Once again, the metric

- This is just the opposite to the Euclidean geometry, where the shortest distance is the straight line. Why?
- In Euclidean metrics, the distance is:

$$d^2 = x^2 + y^2 + z^2$$

- In the spacetime, the proper (Rocket) time is:

$$\tau^2 = (\text{interval})^2 = (\text{earth time})^2 - (\text{earth distance})^2$$

- The Rocket proper time is thus shorter than the Earth time. The Earth moves along the spacetime in a straight line, as do all inertial frames. The Rocket makes a turn: it's dx/dt – the velocity – changes sign! It behaves as a non-inertial frame during that time, and this distinguishes the Rocket frame from the Earth frame.

The “Doppler shift” explanation

• Let’s assume that both twins have extremely powerful telescopes, and they could observe each other’s clock all the time. The clock work by emitting a flash of light every second (of their respective proper time).

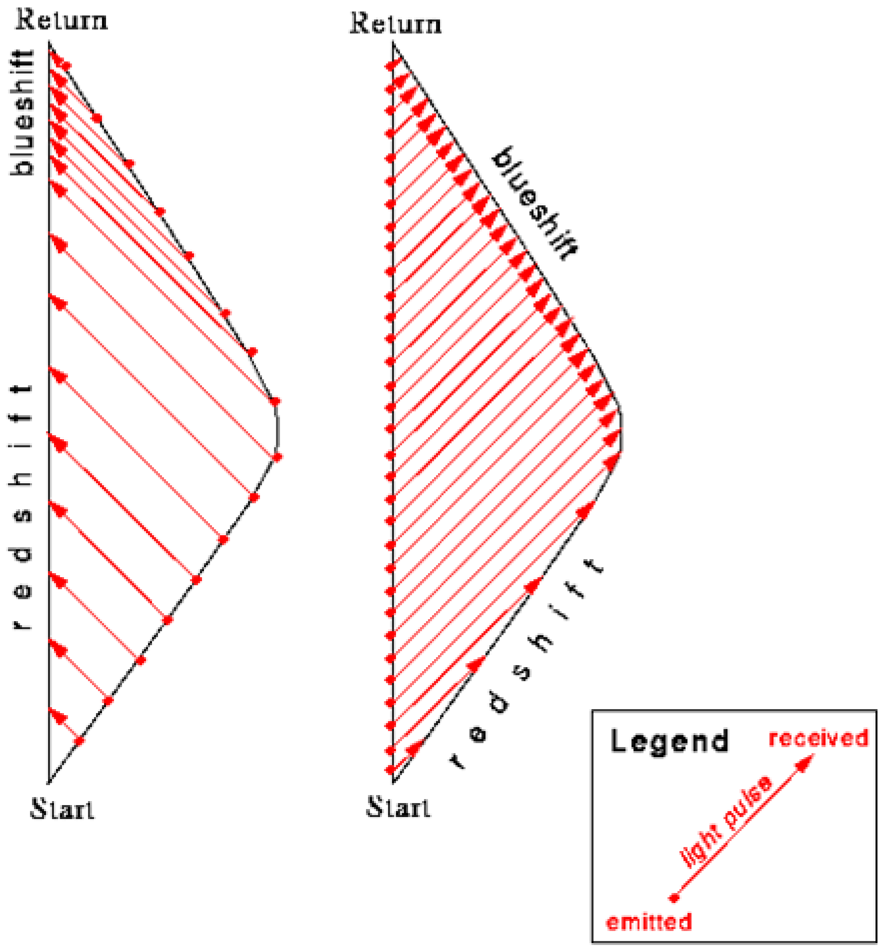
• The what will the Rocket twin see? Recall the Doppler shift formula from the homework. The frequency of light flashes as measured by the Earth twin (and by the Rocket twin) is *reduced* by the factor of $[(1 - v)/(1 + v)]^{1/2}$ on the *outbound* trip, and *increased by the factor of* $[(1 + v)/(1 - v)]^{1/2}$ on the *inbound* trip. So, is everything the same for both twins then? NO!

• The Rocket twin has *less time* to send out the pulses! When counted by the Earth twin, these result in less aging for the Rocket twin. On the other hand, the Earth twin has *more time* to send pulses, so when counted by the Rocket twin, these pulses correspond to more aging of the Earth twin.

Confusing? Let’s look at the spacetime diagram!

The "Doppler shift" explanation

In Earth frame   In Rocket frame



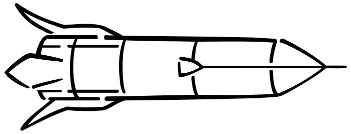
Satisfied? Well, there's more to it...

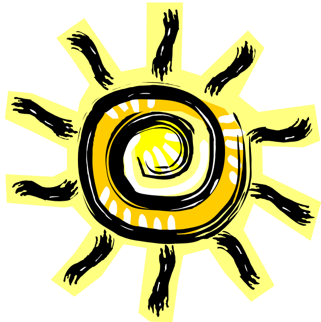
• Let's carefully consider what happens in both frames – the Earth and the Rocket.

• The Earth twin would see the spaceship flying at about $\boldsymbol{v}$ = 0.98c, passing by Earth at x = 0 and t = 0, then by Canopus at x = 99 and t = 101 (times and distances in years). The Earth twin is 101 years older when the spaceship reached Canopus.

• Due to length contraction, the spaceship appears shorter by the $1/\gamma$.

• But, apart from pure curiosity, this doesn't appear important...



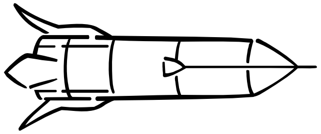


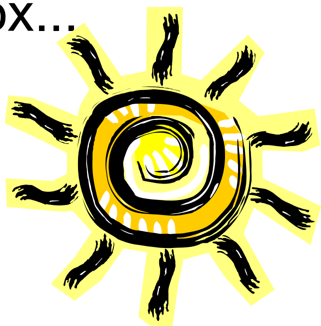
In the Rocket frame

• In the Rocket frame, the Earth, along with Canopus, is flying at 0.98c. To the Rocket twin, the Earth zooms by at this high speed and goes away. Some time later the Canopus zooms by. The Earth by that time is far away. But how far?

• Due to Lorentz length contraction, the Earth, Canopus **and** the distance between them are all shrunk by the factor of $1/\gamma$. Now, this is important! The distance between Earth and Canopus, the distance that the Earth twin travels in the Rocket frame, is only $99/\gamma \approx 19.6$ light years!

• Thus, to the Rocket twin the from Earth to Canopus is only 19.6 light years, which at the speed of 0.98c can be done in just under 20 years! The Rocket twin will be only 20 years older upon arrival to Canopus! The Lorentz contraction alone can resolve the Twin Paradox...





...but what about time dilation?

- The Earth twin sees the Rocket time going slower. It is going slower by the same factor γ, so the 101 years of the Earth time are only 20 years of Rocket time.
- Now let’s hop into the Rocket frame. We’ve seen that the Earth has to travel just about 19.6 light years, which takes about 20 years of Rocket time. Yet, for the Rocket observer the Earth time is going slower by the factor γ. So when the Earth is 19.6 light year away, the Earth clock as seen by the Rocket observer reads $20/\gamma = 3.96$ years! The Earth twin ages less!!!
- Haven’t we just undone the solution for the twin paradox???
- The answer is in carefully looking in what the two twins *see* and what actually *is*.

... AND THAT WE SHALL SEE IN THE NEXT LECTURE