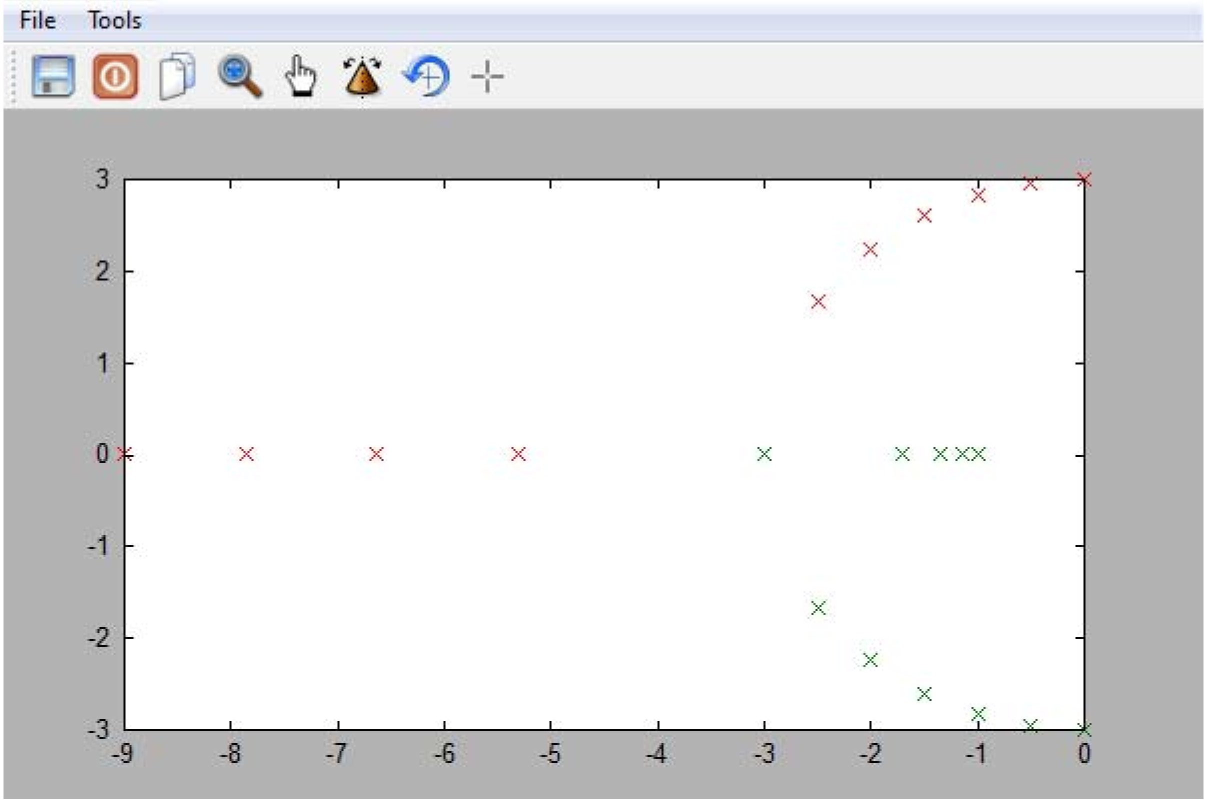
Homework 2 solution

Problem 2: Step response of undamped second-order system with a derivative gain controller.

(a)



The problem asks for each trajectory to be plotted in one color. However, do they really follow one path or another? When the arrive at equal roots (critical damping) there is really no upper or lower root, so they fuse into one. You could have the upper root go left or right, it's up to you.

(b) The system is critically damped at Gd = 6 because the denominator of Y/X has two equal factors.

(c) As Gp $\rightarrow\infty$, the system tends toward conditional stability. The system never becomes unstable as long as Gp is positive. If Gp is negative then it is effectively positive feedback and one would expect an unbounded output y(t).

(d) Long answer…

The transfer function of the system that we are controlling is:

$$H(s) = \frac{1}{s^2 + 9}$$

If we use a PID controller with only the differential component, the answer in problem 1 reduces to….

$$\frac{Y(s)}{X(s)} = \frac{G_D s^2}{s^3 + G_D s^2 + \omega_o{}^2 s} = \frac{G_D s}{s^2 + G_D s + \omega_o{}^2}$$

When the input is a unit step, X(s)=1/s

$$Y(s) = \frac{G_D}{s^2 + G_D s + \omega_o{}^2}$$

Let's write this in the generic form for a signal or transfer function like this:

$$Y(s) = \frac{G}{s^2 + 2\zeta\omega_o s + \omega_o{}^2} = \frac{G}{(s+\zeta\omega_o)^2 + \omega_o{}^2 - (\zeta\omega_o)^2} = \frac{G}{\omega_d}\frac{\omega_d}{(s+\zeta\omega_o)^2 + \omega_d{}^2}$$

where $\omega_d = \omega_o\sqrt{1-\zeta^2}$

The inverse Laplace transform of this expression is

$$y(t) = G\omega_d{}^{-1} e^{-\zeta\omega_o t}\sin(\omega_d t)$$

Now we plug in the values from the problem.

$$G_D = 4, \quad \zeta\omega_o = 2, \quad \omega_o = 3 \quad \zeta = 2/3$$

$$\omega_d = 3\sqrt{1-\frac{4}{9}} = \sqrt{5} = 2.236$$

$$y(t) = \frac{4}{\sqrt{5}} e^{-2t}\sin(\sqrt{5}t)$$

From here on, I will normalize the amplitude to one by ignoring the 4/sqrt(5); you may multiply it in later if you like, but it does not affect any of the percentages or times.

Peak time, percent overshoot and settling time for a step response that actually jumps up to a new value (does not return to zero) are…

$$T_p = \frac{\pi}{\omega\sqrt{1-\zeta^2}} = \frac{\pi}{\sqrt{5}} = 1.405$$

$$\%OS = 100 \times e^{-(\zeta\pi/\sqrt{1-\zeta^2})} = 100 \times e^{-2.81} = 6\%$$

$$T_s = 4/\zeta\omega = 2$$

From my MATLAB plot I estimate that the settling time is near 2 sec. However, I find 0.35<Tp<0.40, and %OS is 35% on the first peak and -2.1% on the first dip.

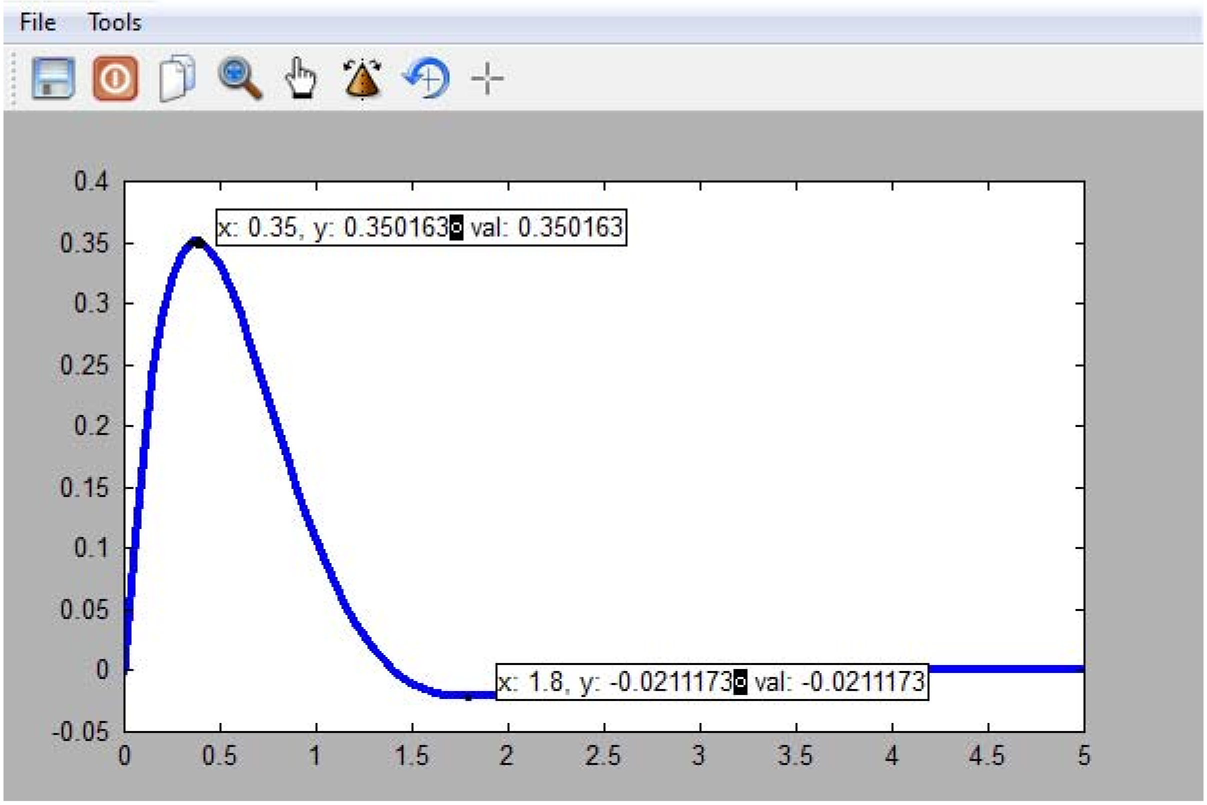
A more appropriate formula for peak time in a response that returns to zero is this:

$$T_p = \frac{\tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)}{\omega\sqrt{1-\zeta^2}} = 0.376\,\text{sec}.$$ You are not expected to know this for the homework.

The peak value, as percentage of a unit step, can be found by plugging this time into y(t):

$$\%OS = 100 \times e^{-2(.376)} \sin\left((\sqrt{5})(.376)\right) = 35\%$$

which matches the plot pretty well.



The plot that we have here is equivalent to a unit impulse response of a standard second order system, such as a series RLC circuit. However, because of the derivative term in the controller, there is an extra "s" factor in the numerator. When the input is a unit step, the LT of that step (1/s) cancels out the "s" so there is no 1/s term after we do partial fractions.

You might be wondering why I wrote the problem to include just the derivative gain, when it gives an answer that does not look like our standard step response. The reason is to show a way to control a system with a purely sinusoidal impulse response; in this problem $h(t)$ is sinusoidal because $H(s) = 1/(s^2+9)$. The most common gain to include in a PID-type controller is proportional. However, try using only proportional gain (Gp) instead of Gd and see what happens to the stability by making a new root locus. Does it give you the desired behavior?