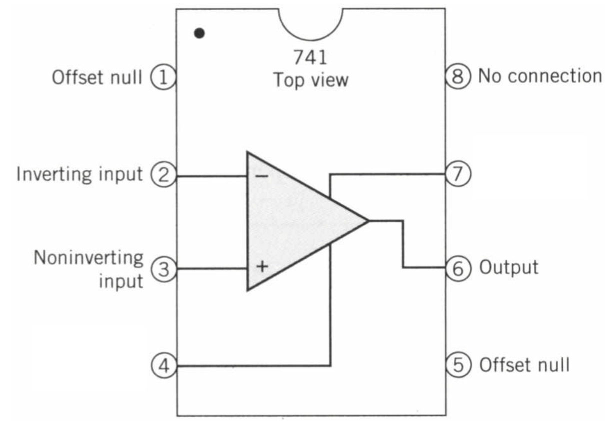
Final Exam

December 12, 2003

1. (16) The following diagram shows a standard op-amp, with the same pin arrangement as a TL081. Mark each of the statements below as:
   A – True (or possible) only in an ideal op-amp
   B – True for real op-amps and ideal op-amps
   C – False for properly functioning op-amp



_A_ $v_2 = v_3$ *(1 pt for B)*

_C_ $v_7 < v_6 < v_4$ *($v_4 < v_7$; 1-pt problem)*

_C_ $v_7 = v_4$ *($v_4 = -v_7$)*

_A_ $i_2 = 0$ *(B also accepted)*

_C_ $i_3 = 10$ mA *(usually i_2, $i_3 < 10$ mA)*

_C_ i_6 is independent of R_L *(i_6 varies to keep v_6 constant)*

A or B v_6 is independent of R_L

_C_ $v_2 > v_3$ and $v_6 = v_7$ *(If $v_2 > v_3$ then $v_6 \gg v_4$)*

2. (16) Assume that cells are cultured on a surface that includes a flat, metal electrode that is about the same size as a cell. The ground electrode is some distance away in the growth medium. We wish to use the potential difference between these electrodes to measure the membrane potential of the cell.
What factors would affect the electrical signal measured by the electrode? Explain briefly how or why each factor changes the reading.

This list includes 34 possible points.

Up to 4 pts each for these factors:

- *Cell-to-electrode spacing (the underside of cells is not flat)*
- *Area of contact (intimate contact only at focal adhesions; total coverage affected by number of cells, size of electrode, cell alignment).*
- *Cell secretions (ions and ECM proteins)*
- *Adhesion of proteins to electrode (charged residues create surface charge; non-polar residues are minor insulators)*

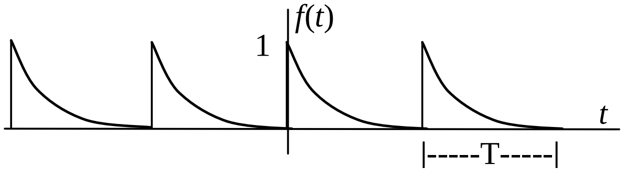
2 pts each:

- *Type of electrode (Pt or AgCl)*
- *Depletion/corrosion of electrode*
- *Distance to ground electrode*
- *Electrical noise/shielding*
- *Redox reactions of solution with electrode*
- *Membrane potential often measured with penetrating electrode*
- *Input impedance of amplifier circuit*

1 pt each:

- *Sampling frequency*
- *Electrode capacitance*
- *Gas bubble formation*
- *Focal adhesions discovered by TIRM*

3. (20) Let f(t) be an infinite train of exponential functions, such that $f(t-nT) = e^{-at}$.
The function f(t) can be represented by a Fourier series with complex coefficients, as shown below.



$$f(t) = C_0 + \sum_{\substack{-\infty \\ n\neq 0}}^{\infty} C_n e^{jnw_0 t}$$

A. Determine the complex coefficients C_0 and C_n for the Fourier series representation of f(t).
B. Plot the approximate magnitude and phase of C_0 and C_n for f(t).

$$C_0 = \frac{1}{T}\int_0^T e^{-at}\,dt = \frac{-1}{aT} e^{-at}\Big|_0^T = \frac{-1}{aT}\left(e^{-aT} - e^{0}\right) = \frac{1 - e^{-aT}}{aT}$$

$$C_n = \frac{1}{T}\int_0^T e^{-at} e^{-jn\omega_0 t}\,dt \quad \text{where } \omega_0 = 2\pi/T$$

$$= \frac{1}{T}\int_0^T e^{-(a+jn\omega_0)t}\,dt$$

$$= \frac{1}{T}\left(\frac{-1}{a+jn\omega_0}\right) e^{-(a+jn\omega_0)t}\Big|_0^T$$

$$C_n = \frac{-1}{T(a+jn\omega_0)}\left[e^{-(a+jn\omega_0)T} - e^{0}\right]$$

note that $\omega_0 T = 2\pi$
and $e^{-jn2\pi} = \cos n2\pi - j\,\cancel{\sin n2\pi}^{\,0}$
$= 1$

$$\text{So, } C_n = \frac{-1}{aT + jn2\pi}\left[e^{-aT} - 1\right]$$

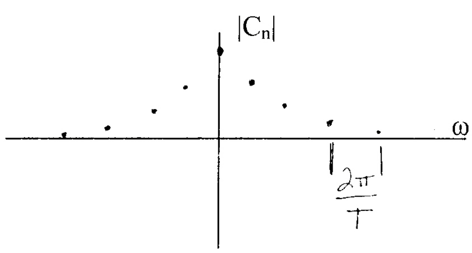
$$\therefore\ C_0 = \frac{1 - e^{-aT}}{aT}$$

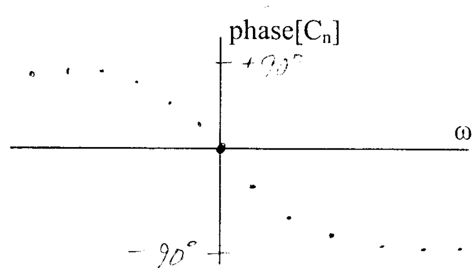
$$C_n = \frac{1 - e^{-aT}}{aT + jn2\pi}$$

Let $A = 1 - e^{-aT}$

$C_0 = A/aT$

$C_1 = A/(aT + j2\pi)$





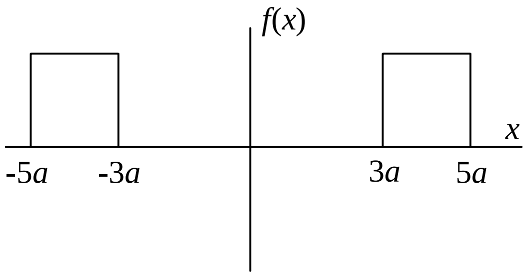
$$|C_n| = \frac{A}{\sqrt{a^2T^2 + 4\pi^2 n^2}}$$

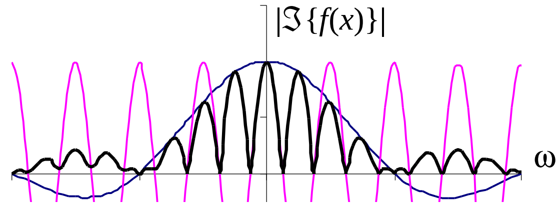
2

$$-\phi_n = \tan^{-1}\left[\frac{n2\pi}{aT}\right]$$

negative because $aT + j2\pi$ is in denominator

4.A (10) Let $f(x)$ be a function consisting of two pulses of height 1. Find its Fourier transform, $\Im\{f(x)\}$, using the convolution property. Draw an approximate plot of $|\Im\{f(x)\}|$.

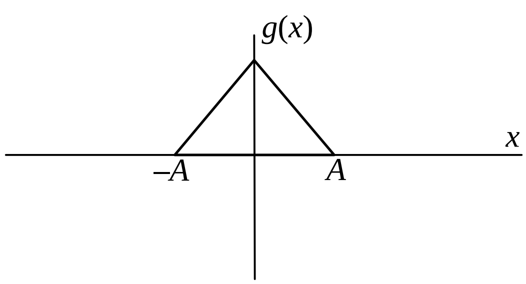


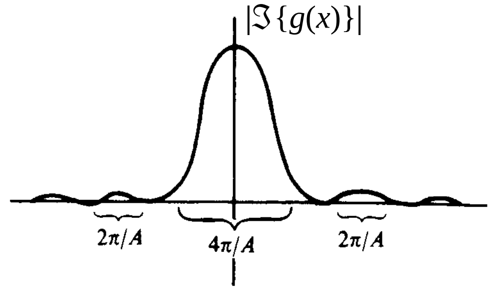


Plot of sinc(aω), cos(4aω), and |sinc(aω)·cos(4aω)|.
Amplitudes are listed below.

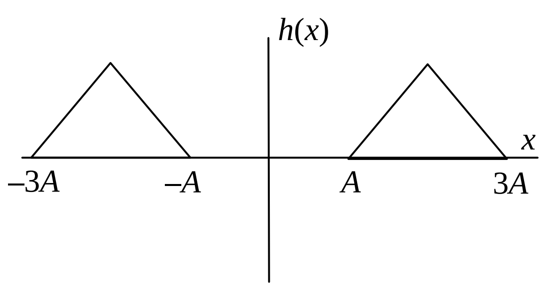
If $p(x)$ is a pulse of height 1, from $-a$ to $+a$, then $f(x)$ is the convolution of $p(x)$ with two impulses of unit area: $f(x) = p(x) \otimes [\delta(x+4a) + \delta(x-4a)]$.
Convolution in space is dual with multiplication in frequency:
$\Im\{f(x)\} = \Im\{p(x)\}\Im\{\delta(x+4a) + \delta(x-4a)\}$ = 2a·sinc(aω)·2cos(4aω)
The sinc function has a height of 2a because the pulse has area 2a.
The cosine has amplitude 2 because there are two pulses of unit area.
The cosine reaches zero when 4aω = nπ, ie when ω = nπ/4a.
The sinc function reaches zero when aω = nπ, ie when ω = nπ/a (except for ω=0).

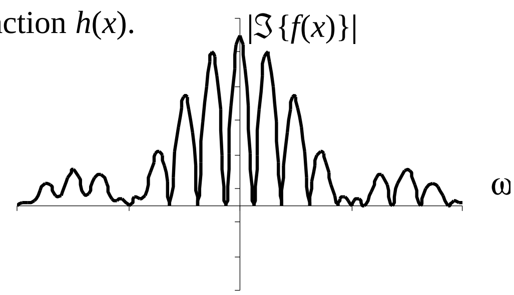
4.B (8) Given the Fourier transform of the single triangle function $g(x)$,





draw the Fourier spectrum of the double-triangle function $h(x)$.





What is the general form of the mathematical expression for $|\Im\{g(x)\}|$?
Answer: sinc^2.
Each triangle is the covolution (in space) of two square pulses. The FT of each pulse is a sinc function, so the FT of the convoluted pulses is sinc()·sinc(). Note that each pulse is half as wide as the triangle, so the frequency of the cosine is 4 times the freqency of the sinc function.
The sinc function equals zero at nπ/2A; the cosine equals zero at 2nπ/A.

5. (25) In the projection of an object through a circular lens:

- Using concepts of Fourier analysis, discuss what we mean when we say that the lens aperture acts as an "optical filter". (10 pts)

Main point: The aperture (either the lens mount or the edge of the lens) is of finite size and diffraction occurs at its edges. The aperture acts as a low-pass filter even if the object does not create a diffraction pattern (e.g., a point source produces no pattern).

When light exits from a pair of slits (for example), three factors combine to create an interference pattern (IP) on a screen: 1) the propagation of light away from the slits, 2) the lateral spreading (diffraction) of light; and 3) the actual constructive or destructive addition of waves at the screen. At the proper distance from the object, the IP has the same shape as the Fourier spectrum of the signal/object. The IP depends on the spacing of features in the light source (spatial frequencies). The complete pattern extends laterally to infinity, with high spatial frequencies causing diffraction through the widest angle.

An image is formed when a lens refocuses the diffracted rays. In essence, the image is an interference pattern of the original interference pattern. Propagation of light from the lens to the image plane has the effect of an inverse Fourier transform (IFT). If an aperture restricts the width of the lens, it allows the lens to focus only part of the IP as the image. Some spatial frequencies are rejected, so the image looks different from the object. The aperture may block any portion of the lens, but every lens has finite width so the highest spatial frequencies are always rejected.

Our optical filter is the product of the round aperture (1 for $r<R$, 0 elsewhere) and the 2D frequency spectrum. Therefore, the lens and aperture create an image that is the convolution of the original object with the IFT of the aperture (e.g., point source → Bessel function).

- Discuss whether the lens passes mainly low, high, or a band of spatial frequencies. (5)

It is a low-pass filter. The optical axis of the lens defines the $\lambda=0$ axis of the IFT; λ increases with distance from the optical axis. The aperture includes all of the low-frequency region and the lens mount blocks all of the high-frequency region.

- Discuss the implications that this filtering has on the ultimate resolving capability that an instrument based on this lens can achieve. (10 pts)

Resolution is a measure of how close two points in an object can be and still be distinguished as two points in the image. We can distinguish two points when their intensity profile consists of a peak for each one and a contrasting region between them.

Ideally the image of each point source is also a point, i.e. a delta function, that can be distinguished from another delta function at almost zero distance. Note that $\Im\{\delta(x-x_0)\}$ is constant over all frequencies, and that sharp edges in a sawtooth or square wave require a F. series with a (nearly) infinite number of terms. Our lens aperture rejects high frequencies, so its FT is a Bessel function. The narrower the aperture, the wider/flatter the Bessel function, and the farther apart two points must be so their images do not blur together.

Generally, using a wider lens – one that focuses light into the image from a wider angle – improves the ultimate resolving capability of the system. A wide lens does not guarantee good resolution: the wavelength of light creates another limit, and wider lenses are more subject to chromatic and spherical aberration, as well as manufacturing flaws.

[This is one possible answer; the order and details in your answer(s) may vary.]