`Final Exam Solutions

December 13, 2004

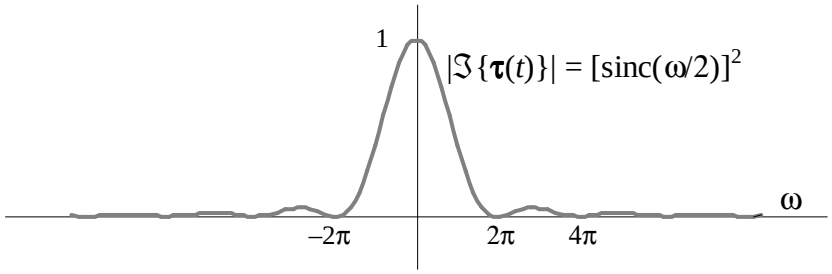
1. Fourier transforms and convolution [10 points each for (a) and (b), 30 points for (c)].

(a) Let $p_1(t)$ be a square pulse with A=1 and δ=1, such that $P_1(\omega) = \dfrac{\sin(\omega/2)}{\omega/2}$.

The triangle is the convolution of two pulses: $\tau(t) = p_1(t) \otimes p_1(t)$.

The convolution property states that $\Im\{f_1(t) \otimes f_2(t)\} = \Im\{f_1(t)\}\Im\{f_2(t)\}$, so ...

$$T(\omega) = \Im\{\tau(t)\} = \Im\{p_1(t) \otimes p_1(t)\} = \Im\{p_1(t)\}\Im\{p_1(t)\} = \{P_1(\omega)\}^2 = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^2 = \text{sinc}^2(\omega/2).$$



b) Doubling the width of $\tau(t)$ means scaling time by 0.5, i.e. $\tau_2(t) = \tau(t/2)$, while keeping the height of the triangle constant.

Using the inverse time/frequency scaling property, $f(at) \Leftrightarrow \dfrac{1}{a} F\left(\dfrac{\omega}{a}\right)$ make a FT pair for a>0.

Therefore, $T_2(\omega) = 2T(2\omega) = 2\text{sinc}^2(\omega)$. <u>The new plot is twice as tall but half as wide</u>. Note that if the *pulse* were doubled in width, the triangle would be twice as wide and twice as tall, so the new plot would be four times as high.

(c) The signal $q(t)$ is the convolution of $\tau(t)$ with the sum of four delta functions:

$q(t) = \tau(t) \otimes [-¼\delta(t+20) + \delta(t+2) - \delta(t-2) + ¼\delta(t-20)]$

By the convolution property, $Q(\omega) = T(\omega) \cdot \Im\{[-¼\delta(t+20) + \delta(t+2) - \delta(t-2) + ¼\delta(t-20)]\}$.

From part (a), we know that $T(\omega) = \text{sinc}^2(\omega/2)$.

From our FT table, we know that $\Im\{\delta(t-a)\} = e^{-j\omega a}$. This is also an example of the time shift property, where a = +/–2 or +/–20.

The FT is a linear operation, so $\Im\{¼\delta(t-20)\} = ¼\, e^{-20j\omega}$.

So, $\Im\{[-¼\delta(t+20) + \delta(t+2) - \delta(t-2) + ¼\delta(t-20)]\} = -¼e^{20j\omega} + e^{2j\omega} - e^{-2j\omega} + ¼e^{-20j\omega}$

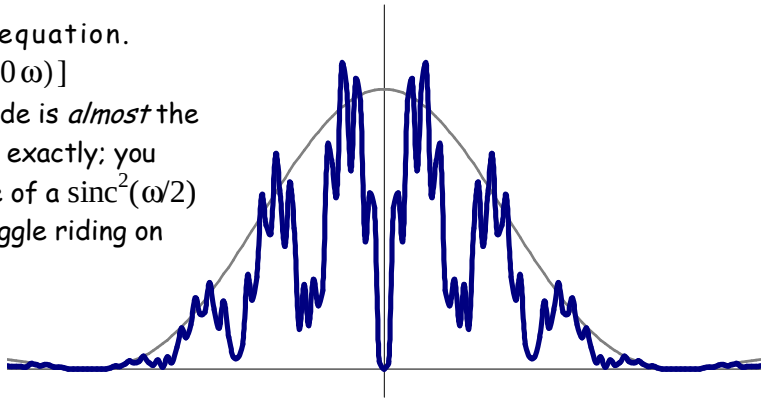
$= [e^{2j\omega} - e^{-2j\omega}] - ¼[e^{20j\omega} - e^{-20j\omega}]$

$= 2j\sin(2\omega) - ¼[2j\sin(20\omega)]$ by Euler's equation.

$\therefore\ Q(\omega) = 2j \cdot \text{sinc}^2(\omega/2)[\sin(2\omega) - ¼\sin(20\omega)]$

Note that $Q(\omega)$ is all imaginary, and its magnitude is *almost* the same as Im{ $Q(\omega)$}. This is complicated to plot exactly; you should show that there is an envelope the shape of a $\text{sinc}^2(\omega/2)$ curve, with a sine wave inside it and a small squiggle riding on the larger sine wave.

[20 points for $Q(\omega)$, 10 points for the plot]



2. (50) In this problem you are asked to explain how darkfield microscopy works.

a-c) The optical system is shown on page 2 of section 9 in the course notes. The two most important light paths are the undiffracted light (which originates at the annular aperture and does not enter the objective lens) and the diffracted/refracted light, which does enter the objective lens. The function of each element can be explained as follows:

- The **annular aperture** blocks the middle disk of light (around the center of the optical axis) so only a ring of light enters the condenser.
- The **condenser** focuses the light from the aperture into the specimen plane. The light comes in from an angle but the annulus itself is not visible in the specimen plane.
- The **image plane** is the location where diffracted light from the objective is reconstructed as an image of the object. It can be a plane in space or a screen where the image is displayed. The eyepiece focuses on this plane, and creates another image plane; our eyes then focus on (but are not located at) this second image plane.
- The **objective lens** receives light diffracted/refracted by the specimen, magnifies the image, and focuses it (at 160 mm or at infinity, depending on the objective).
- The **source** emits the light that illuminates the specimen.
- The **specimen plane** is the location of the object that we wish to see. The condenser and the objective lens both focus here.

d) Why is this technique called "darkfield"? [4 points]

Small specimens appear bright on a dark background because light reaches our eyes only via refraction or diffraction. When there is no specimen, the field of view is entirely dark. It is the opposite of brightfield microscopy, in which specimens appear dark on a bright background.

e) 1) [2 points] A darkfield microscope effectively acts as a band-pass filter.

2) [8 points] Spatial filtering depends on diffraction of light by the specimen. The angle of diffraction increases as the spatial frequency <u>in the specimen</u> increases (i.e., as the spacing between features decreases). When we capture light that has been diffracted a moderate amount, we observe the middle range of spatial frequencies; when we block light that has diffracted through only a small angle (or not at all), we reject the low frequency and DC components.

In darkfield microscopy the annular aperture plus the condenser provide a narrow, angled cone of light. This zero-order light misses the objective, creating the high-pass (low-reject) aspect of the filter. Some of the diffracted light enters the objective. When the diffraction angle is too small, it misses with the zero-order light; when the diffraction angle is too large, it misses over the other side of the objective. Therefore, it is a band-pass system. [In contrast, if our finite objective captured all of the zero-order light through an annular aperture, the system would be a low-pass spatial filter.]

Band-pass filtering means that darkfield microscopy enhances the contrast of edges in the specimen. As a result, the center of a large opaque specimen and the center of a large transparent specimen will both appear dark (low frequencies are rejected) and the edges will not be detected with perfect resolution (high frequencies are rejected, too).

[Some answers confused the dependence of diffraction angle on the wavelength of light with the dependence on the feature size in the sample. We are interested in the spatial frequencies <u>in the sample</u>, e.g. how close together two bacteria are.]