Problem 2: Pendulum

Part (a), Write differential equation

$$ma = \sum F_\theta$$

$$mL\ddot{\theta} = -mg\sin(\theta) - \beta L\dot{\theta} + \frac{x(t)}{L}$$

$$\ddot{\theta} + \frac{\beta}{m}\dot{\theta} + \frac{g}{L}\sin(\theta) = \frac{x(t)}{mL^2}$$

$$\frac{\beta}{m} = \frac{5}{1.2} = 4.2\,\mathrm{sec}^{-1}, \quad \frac{g}{L} = \frac{10}{.4} = 25\,\mathrm{sec}^{-2}, \quad mL^2 = 1.2\cdot .4^2 = .192\,\mathrm{kg\ m}^2$$

Part (b), write as system of first order <u>non-linear</u> differential equations

Let $y_1 = \theta$

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -\frac{g}{L}\sin(y_1) - \frac{\beta}{m}y_2 + \frac{x(t)}{mL^2}$$

Part (c), Find equilibrium points

Let $x(t) = 0$

$$\dot{y}_1 = 0 \rightarrow y_2 = 0$$

$$\dot{y}_2 = 0 \rightarrow -\frac{g}{L}\sin(y_1) - \frac{\beta}{m}(0) = 0 \rightarrow \sin(y_1) = 0 \rightarrow y_1 = n\pi$$

Eq. points are $(n\pi, 0)$

Part (d), linearize about (π,0) and write in state space (matrix) form
From

$$\begin{bmatrix} \frac{\partial \dot{y}_1}{\partial y_1} & \frac{\partial \dot{y}_1}{\partial y_2} \\ \frac{\partial \dot{y}_2}{\partial y_1} & \frac{\partial \dot{y}_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\sin(y_1) & -\frac{\beta}{m} \end{bmatrix}_{\pi,0} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & -\frac{\beta}{m} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 25 & -4.2 \end{bmatrix} \equiv A$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & -\frac{\beta}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/mL^2 \end{bmatrix} x(t), \quad \psi = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Note that the y variables in the second line have become equilibrium variables, that is they are relative to (π,0), not (0,0). Let's pretend that they are written as y_{e1} and y_{e2}.

Part (e), find transfer function $H(s)=\psi(s)/X(s)$

$$s\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & -\frac{\beta}{m} \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/mL^2 \end{bmatrix} X(s)$$

$$[sI-A]\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/mL^2 \end{bmatrix} X(s)$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} s & -1 \\ -\frac{g}{L} & s+\frac{\beta}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1/mL^2 \end{bmatrix} X(s) = \frac{\begin{bmatrix} s+\frac{\beta}{m} & 1 \\ \frac{g}{L} & s \end{bmatrix}}{\begin{vmatrix} s & -1 \\ -\frac{g}{L} & s+\frac{\beta}{m} \end{vmatrix}} \begin{bmatrix} 0 \\ 1/mL^2 \end{bmatrix} X(s) = \frac{mL^2\begin{bmatrix} 1 \\ s \end{bmatrix}}{s^2+\frac{\beta}{m}s-\frac{g}{L}} X(s)$$

$$\Psi(s) = [1 \quad 0]\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{mL^2[1 \quad 0]\begin{bmatrix} 1 \\ s \end{bmatrix}}{s^2+\frac{\beta}{m}s-\frac{g}{L}} X(s) = \frac{mL^2}{s^2+\frac{\beta}{m}s-\frac{g}{L}} X(s)$$

$$\frac{\Psi(s)}{X(s)} = \frac{mL^2}{s^2+\frac{\beta}{m}s-\frac{g}{L}} = = \frac{.192}{s^2+4.2s-25} \text{rad/N}\cdot\text{m}$$

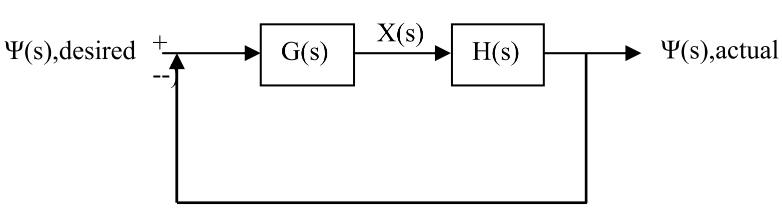
Note that if you did not have $x(t)/mL^2$ as your forcing term, you might not have the 0.192 in the numerator.

Part (f), find roots, determine type of eq point and stability

$$\lambda = -2.1 \pm \sqrt{4.41+25} = -2.1 \pm 5.4 \rightarrow 3.3, -7.5$$

Real roots of opposite sign, saddle point, unstable.

Part (g), diagram



Note that X(s) is the torque that is regulated by the controller and passed into the pendulum system. Therefore it is between G and H.
Some of you put G in the feedback path. This is a legitimate controller, but it does not give quite the same equations as we usually see.

Part (h), proportional controller

For controller in forward path,

$$\frac{\Psi_{OUT}}{\Psi_{IN}} = \frac{G}{G - 1/H}$$

On homework you have derived the general form of the PID controller transfer function:

$$\frac{\Psi_{OUT}}{\Psi_{IN}} = \frac{A(K_D s^2 + K_P s + K_I)}{s^3 + (AK_D + b)s^2 + (AK_P + c)s + AK_I}$$

Which is true when

$$H(s) = \frac{A}{s^2 + bs + c}$$

where A is some constant, not the state matrix. I have avoided the natural frequency variable ω because this is a saddle point and it does not oscillate.

You would then cancel as many K terms as necessary for your particular type of controller (P, PD, PI, or PID).

For G=Kp we get:

$$\frac{\Psi_{OUT}}{\Psi_{IN}} = \frac{AK_P}{s^2 + bs + (AK_P + c)}$$

For our transfer function H(s), this results in…

$$\frac{\Psi_{OUT}}{\Psi_{IN}} = \frac{.192K_P}{s^2 + 4.2s + (.192K_P - 25)}$$

For critical damping, we want the denominator to have repeated roots.

$$\frac{\Psi_{OUT}}{\Psi_{IN}} = \frac{.192K_P}{s^2 + 4.2s + 2.1^2} = \frac{.192K_P}{s^2 + 4.2s + (.192\,K_P - 25)}$$

$$2.1^2 = .192K_P - 25 \rightarrow K_P = 153\,\mathrm{N \cdot m/rad}$$

Note the units for K_P; the error is in radians, and the forcing function $x(t)$ is torque.

If you omitted the $A=mL^2$ factor in the $x(t)$ term in part (a) of this problem, your answer should be Kp=29.4 Nm/rad.

Part (i), PD controller

To set up this part of the problem, we should define the damping ratio and natural frequency of the complete system as follows:

$$\frac{\Psi_{OUT}}{\Psi_{IN}} = \frac{something}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Note that ζ and ω do not refer to H(s) because H(s) is not a damped oscillatory system; see part (h) for its notation.

According to the formula for a PID-controlled system, we see that

$$AK_D + b = 2\zeta\omega, \quad AK_P + c = \omega_0^2$$

The settling time condition is the simpler of the two to solve. The problem states the settling time is 1.25 seconds. By the formula given on the exam,

$$t_s = \frac{4.6}{\zeta\omega_0} \rightarrow \zeta\omega_0 = \frac{4.6}{1.25} = 3.68$$

It is easy to solve for Kd:

$$K_D = \frac{2\zeta\omega - b}{A} = \frac{2(3.68) - 4.2}{A}$$ = 3.16 if you let *A*=1, or 16.4 if *A*=0.192.

You can't solve for Kp unless you make some other assumption, like the system is critically damped, or the overshoot is given.

If you assume the system is critically damped, then ζ=1, ω=3.68, and

$$Kp = \frac{\omega_0^2 - c}{A} = \frac{3.68^2 + 25}{A}$$, which is 38.5 if *A*=1, and 201 if *A*=.192.

If, on the other hand, you found the damping ratio from the overshoot specification, you could also solve for Kp.

The overshoot calculation is ugly, and requires you to solve for the damping ratio ζ. In the following equation, *OS* stands for the proportional overshoot, i.e. 100*%*OS*. My calculations give…

$$\zeta = \sqrt{1 + \frac{\pi}{(\ln|OS|)^2}} = \sqrt{1 + \frac{\pi}{(\ln|.05|)^2}} = 1.16$$

If we now find ω=3.68/1.16=3.17, then Kp=35/*A*.

Another way to use the overshoot information is to ignore the settling time and keep the Kp value from part (h). Note that we cannot use the critical damping condition because the damping is not critical (*i.e.* ζ≠1). We start with the equation for a PD controller:

$$\frac{\Psi_{OUT}}{\Psi_{IN}} = \frac{A(K_D s + K_P)}{s^2 + (AK_D + b)s + (AK_P + c)}$$

Here we know *A*, *b*, *c,* and Kp=153. We cannot solve for both KD and Kp with one equation. We cannot use the

$$K_D = \frac{2\zeta\sqrt{AK_P + c} - b}{A} = \frac{2(1.16)\sqrt{153A - 25} - 4.2}{A}$$

If you choose *A*=.192, Kd = 3.5. If you left *A*=1, I think you would have found Kp=29.4, giving you and Kd=.67 here. I'm not sure.

Of course, so few people got past part (a) with the same numbers I did that the values provided above don't mean much as an example solution. The result:

$$\frac{\Psi_{OUT}}{\Psi_{IN}} = \frac{.67s + 29.4}{s^2 + 4.9s + 4.41}$$

 Name Solution

Part (j), steady-state error

Substitute jω for *s*.

$$\frac{\Psi_{OUT}}{\Psi_{IN}} = \frac{.67\,j\omega + 29.4}{(j\omega)^2 + 4.9\,j\omega + 4.41}$$

The steady state error refers to the gain when ω=0, so set ω=0 in the equation. The result is 29.4/4.41 = 6.67. This is a ridiculously large error and makes me wonder whether everything else in this problem is right, but there it is…

Part (k), steady-state error

To eliminate the steady-state error, we would include an integral gain, K_I.

Part (l), its effect

Integral gains usually increase the overshoot.