3. (30 points) Riding on a city bus presents an interesting biomechanical problem. Road roughness is passed through the suspension into the seats, and into the spine and head of the traveler. Analysis of the spine is similar to analysis of the bus suspension, but the bus is simpler so let's save the spine for a homework problem. Model the bus as a mass m riding on a spring k and a damper b; the spring and damper are in parallel between the bus and the wheel. The wheel follows the contours of the road.

a) Draw a schematic of the mass-spring-damper system. Show the position of the bus as $y(t)$ and the height of the road as $x(t)$, both defined positive upward. The position of the bus is the output, and position of the ground is the input.

The mass (bus) should be sitting atop the spring and damper, which are side by side. x(t) and y(t) are defined upward from the same level, such that y-x is the length of the damper and also the length of the spring. When y-x gets larger, the spring pulls down more. When d(y-x)/dt is positive, the damper is pulling down on the mass.

b) Create a transfer function that relates the height of the bus to the height of the ground. You may start with impedances or with a differential equation.

The easiest way is from the differential equation.

$$ma = \sum F_{up}$$
$$m\ddot{y} = -k(y-x) - b(\dot{y}-\dot{x})$$
$$m\ddot{y} + b\dot{y} + ky = b\dot{x} + kx$$

Then take the Laplace transform…

$$ms^2Y + sbY + kY = sbX + kX$$
$$\frac{Y}{X} = \frac{sb+k}{ms^2+bs+k}$$

You could also remap the mechanical system by placing a capacitor in series with a resistor, and putting them both in parallel with an inductor. The parallel paths are then driven by a current source, analogous to dx(t)/dt. You would then use the impedances to calculate the current through the inductor. The resulting transfer function would actually be sY/sX because both values are velocities (currents). However, the s's cancel and we are left with the transfer function Y(s)/X(s).

[Part B, continued]

Many of you correctly used parallel / series impedance relationship:

$$Z_{TOTAL,X} \equiv \frac{F_X}{V_X} = \frac{ms(b+\frac{k}{s})}{ms^2+b+\frac{k}{s}} = \frac{ms(bs+k)}{ms^2+bs+k}$$

However, the impedance is the relationship between force F and velocity V. The total impedance above applies at x(t), i.e. at the wheel. You can also find the impedance of the mass, which is very simple:

$$Z_{TOTAL,Y} \equiv \frac{F_Y}{V_Y} = ms$$

Now, note that the force at x is the same at y; all of the force has to be transmitted through the spring and damper.[1] Therefore, we can divide the two impedances…

$$\frac{V_Y}{V_X} = \frac{F}{V_X} \div \frac{F}{V_Y} = \frac{ms(b+\frac{k}{s})}{ms^2+b+\frac{k}{s}} \div ms = \frac{bs+k}{ms^2+bs+k}$$

Of course, we really want Y(s)/X(s), not V_Y/V_X. It's the same:

$$\frac{V_Y}{V_X} = \frac{sY}{sX} = \frac{Y}{X}$$

So we have just found a transfer function that matches what we found with the differential equation.

c) Let k = 400 kN/m, m = 5000 kg, and b = 45000 N-s/m. Find the resulting gain function $G(\omega)$.

Divide the transfer function by m, calculate the constants, plug them into the transfer function, then substitute $s = j\omega$.

$$\frac{b}{m} = \frac{45000}{5000} = 9, \quad \frac{k}{m} = \frac{400{,}000}{5000} = 80$$

$$\frac{Y}{X} = \frac{9s+80}{s^2+9s+80}$$

$$G(\omega) = \frac{9j\omega+80}{(j\omega)^2+9j\omega+80} = \frac{80+9\omega j}{80-\omega^2+9\omega j}$$

[1] Of course, the force in the spring is not equal to the force in the damper, but the force from both of them is all transmitted to the mass.

d) Roughly sketch the magnitude and phase of G(ω) by finding values at ω=0, ω→∞, and other value(s) of ω as needed; one middle value might be enough.

$$G(0) = \frac{80}{80} \Rightarrow |G(0)| = 1, \quad \angle G(0) = 0$$

For infinity, divide top and bottom by ω^2.

$$G(\omega) = \frac{80/\omega^2 + 9j/\omega}{80/\omega^2 - 1 + 9j/\omega}$$

$$G(\infty) \to \frac{0}{-1} \to |G(\infty)| = 0, \quad \angle G(0) = \pm 180^\circ$$

The other good choice will make the denominator all imaginary: $\omega^2 = 80$.

$$G(\omega) = \frac{80 + 9j\sqrt{80}}{80 - 80 + 9j\sqrt{80}} \approx \frac{80 + 80j}{80j} = \frac{1+j}{j}$$

$$|G(\sqrt{80})| = \sqrt{2}, \quad \angle G(\sqrt{80}) = \arctan(1/1) - \arctan(1/0) = 45 - 90 = -45^\circ$$

This suggests that the phase magnitude plot starts at 1, rises to 1.414, and falls to zero: a resonant low-pass filter. The phase plot starts at 0 and falls gradually to –180°.

A plot generated with MATLAB is shown on the following page.

e) In some places the highway has bumps that oscillate the wheels at 4 Hz. What is the magnitude of the gain at 4 Hz?

Remember, $\omega = 2\pi f$. Here, 4 Hz ≈25 rad/sec; $|G(\omega)| \approx 0.4$.

Answer <u>one</u> of the following questions. Logic is good, equations are better.

f_1) If the driver slows down to 75% of the original speed, how does the maximum force experienced by the passengers change?

We are now at 3 Hz, or 18.8 rad/sec. The new gain is about 0.57. It is about 40% higher than before, because we have moved closer to the resonant peak. However, this describes the amplitude of the bus motion. Force is proportional to acceleration, and acceleration is proportional to the square of the frequency for any sinusoidal function. So, the ratio caused by the frequency is $(\frac{3}{4})^2 \approx 0.56$. The net result is 1.4 × 0.56 = 0.78, or a 22% reduction in the force.

 Name Solution

f2) If the bus takes on 20 additional passengers at 80 kg per passenger (a 65 kg human carrying a 15 kg backpack), how does the maximum force experienced by the passengers change?

This new payload increases the mass from 5000 to 6600 kg. Calculate new values, b/m=6.82, k/m=60.6. Substituting in the transfer function from part (b), I get the following transfer function:

$G(\omega) = \dfrac{60.6 + 6.82\omega j}{60.6 - \omega^2 + 6.82\omega j}$. Letting f=4 Hz, so ω=25, I get |G(25)|=0.31 which suggests that the oscillations are smaller with the added weight. The frequency is unchanged from 4 Hz so the acceleration is proportional to the amplitude.

Plot for part (d):