Quiz #4.1

November 28, 2008

Solution

Consider the following second order differential equation, in which $y' = dy/dt$ and $u(t)$ is a user input.

$$y'' - (y')^2 / 10 + 3y'y - y^2 + 4 = 7u(t)$$

1. Convert this differential equation into a system of first-order differential equations.

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = (y_2)^2 / 10 - 3y_2 y_1 + y_1^2 - 4 = 7u(t)$$

2. Find the equilibrium points for this system, and mark them on the axes below. Label the horizontal and vertical axes appropriately.

Set the two derivatives and the user input equal to zero, and solve for y_1 and y_2.

$$\dot{y}_1 = 0 = y_2$$

Plug this into the second equation:

$$\dot{y}_2 = 0 = 0 - 0 + y_1^2 - 4 + 0$$
$$y_1 = \pm 2$$

The two equilibrium points are (2,0) and (–2,0).

3. Linearize the system about each equilibrium point.

Using the derivatives shown in the Jacobian matrix,

$$A = \begin{bmatrix} 0 & 1 \\ -3y_2 + 2y_1 & y_2/5 - 3y_1 \end{bmatrix}$$

Then we evaluate A at each equilibrium point.

For (2,0), $A = \begin{bmatrix} 0 & 1 \\ 4 & -6 \end{bmatrix}$

For (–2,0), $A = \begin{bmatrix} 0 & 1 \\ -4 & 6 \end{bmatrix}$

4. Write each linearized system in state space form, using $v(t)$=8y as the system output.

For (2,0)...

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ 4 & -6\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}+\begin{bmatrix}0\\ 7\end{bmatrix}x(t)$$

$$v(t)=\begin{bmatrix}8 & 0\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}$$

For (-2,0)...

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -4 & 6\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}+\begin{bmatrix}0\\ 7\end{bmatrix}x(t)$$

$$v(t)=\begin{bmatrix}8 & 0\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}$$

5. Find the eigenvalues (roots) for each linearized system.

The roots satisfy the equation $|sI - A| = 0$ where I is the identity matrix.

$$\det\begin{bmatrix}s & -1\\ -4 & s+6\end{bmatrix}=0$$

$$s^2+6s-4=0$$

For (2,0)... $s=-3\pm\sqrt{13}=(0.6,-6.6)$

$$\det\begin{bmatrix}s & -1\\ 4 & s-6\end{bmatrix}=0$$

$$s^2-6s+4=0$$

For (-2,0)... $s=3\pm\sqrt{5}=(0.76,5.24)$

6. State the type(s) of critical points about which you have linearized the system.

The point at (2,0) is a saddle point. The point at (–2,0) is an unstable node. We have not talked about these points enough for you to recognize a node. You should recognize a center (imaginary roots), a spiral (complex roots) and a saddle point (real roots of opposite sign).

7. Make a rough sketch of the system behavior at each point. Use stability information and the sign of *y* and *y*' to determine the general direction of the trajectories. Remember that *y* and *y*' in a linearized system are measured from the critical point, not from the origin of the non-linear system.

8. Optional: Join the trajectories from the two equilibrium points to get an estimate of the system behavior.

I will not expect you to make a sketch like this on your quiz. I have plotted the overall system below. Note that the point (0,0) is at the center of the plot, not at the intersection of the axes.