1a. The first problem can be solved in the time domain with differential equations, or in the frequency domain with complex numbers and phasors.

Time domain solution

Our system has four unknowns: i_1, i_2, i_3, and $v_{out}(t)$, so we write four equations:

$i_1 + i_2 + i_3 = 0$

$v_1 = L\, di_1/dt + i_1R_1$

$v_2 = i_2R_2 \rightarrow i_2 = v_2 / R_2$

$v_{out} = q/C = \int i_3\, dt / C \rightarrow i_3 = C\, dv_{out} / dt = (.02\text{ F})\, dv_{out} / dt$

To find i_1 we need to solve a differential equation. This is easier if we rewrite v_1 first:

$v_1 = 25\cos(5t + 90°) = 25\cos(90°)\cos(5t) – 25\sin(90°)\sin(5t) = – 25\sin(5t)$

$\therefore L\, di_1/dt + i_1R_1 = – 25\sin(5t)$

We want the steady-state solution for the circuit, so we only need the particular solution to the differential equation.

Guess that $i_{1p} = A\cos(5t) + B\sin(5t)$

$L\{–5A\sin(5t) + 5B\cos(5t)\} + R_1\{A\cos(5t) + B\sin(5t)\} = – 25\sin(5t)$

$(R_1A + 5LB)\cos(5t) + (–5LA + R_1B)\sin(5t) = – 25\sin(5t)$

$(R_1A + 5LB) = 0 \rightarrow 3A + 4B = 0$

$(–5LA + R_1B) = – 25 \rightarrow –4A + 3B = –25$

Solving by determinants, inspection, or substitution gives

$A = 4$, $B = –3$ so $i_{1p} = 4\cos(5t) – 3\sin(5t)$.

To find i_2 we need to expand the expression for v_2, noting that $\sqrt{72} = 6\sqrt{2}$.

$i_2 = v_2 / R_2 = 6\sqrt{2}\{\cos(45°)\cos(5t) – \sin(45°)\sin(5t)\} / 6$

$i_2 = \cos(5t) – \sin(5t)$.

Now we can substitute our expressions for i_1, i_2, & i_3 into the equation $i_1 + i_2 + i_3 = 0$:

$4\cos(5t) – 3\sin(5t) + \cos(5t) – \sin(5t) + .02(dv_{out}/dt) = 0$

$dv_{out}/dt = –250\cos(5t) + 200\sin(5t)$

Integrating...

$v_{out} = –50\sin(5t) – 40\cos(5t)$ + a constant, which we ignore*

We want the answer to be in the form $M\cos(5t + \theta)$, which is the same as...

$v_{out} = M\cos(\theta)\cos(5t) – M\sin(\theta)\sin(5t)$

This shows us that $M\cos(\theta) = –40$ and $M\sin(?) = 50$, so $\tan(\theta) = +5/–4$ and θ is in quadrant II.

From the table of tangents, we get atan(5/4) = 51.3°, so atan(–5/4) = 180 – 51.3 = 128.7°

$M = \text{sqrt}(40^2 + 50^2) = 10\sqrt{(16 + 25)}$

$v_{out} = 10\sqrt{41}\{\cos(5t + 128.7°)\}$

* The constant is determined by the time at which the voltage across the capacitor is set to zero. Usually Vout oscillates around some offset voltage that can be larger than the amplitude of Vout itself. A real circuit would have a reset switch that bypasses the capacitor, and only by opening this switch at precisely the right time can the offset voltage be reduced to zero. This offset could be called a transient effect, but it never dies out (in an ideal circuit). You can try it in PSPICE and see.

One of the many frequency domain (phasor) solutions

Assume ideal op-amp behavior, so

$v- = v+ = 0$ because the open-loop gain is infinite and v+ is connected to ground;

$i- = i+ = 0$ because there is no current into the op-amp at the signal inputs.

Apply KCL at the v– node, using capital letters to represent complex currents, $I = I_{MAX}e^{jwt}$

$I_1 + I_2 + I_3 = 0$

Write currents in terms of voltages, using complex impedances

$I_1 = (V_1 - 0) / Z_1$, etc.

Substitute into the sum of currents

$V_1/Z_1 + V_2/Z_2 + V_{out}/Z_3 = 0$

Solve to get the complex gain equation for an inverting summing amplifier

$V_{out} = -V_1 (Z_3/Z_1) - V_2 (Z_3/Z_2)$ Note the minus signs!

Write the input voltages as phasors

$V_1 = 25\cos(5t + 90°) \rightarrow 25 \angle 90°$

$V_2 = \sqrt{72}\cos(5t + 45°) \rightarrow 6\sqrt{2} \angle 45°$

Find the impedance for each of the current paths in rectangular and phasor forms

$Z_1 = Z_{R1} + Z_L = R_1 + jwL = 3 + j(5)(.8) = 3 + 4j\ \Omega$ -or- $5\angle \mathrm{atan}(4/3)$

$Z_2 = R_2 = 6\ \Omega$ -or- $6\angle 0°$

$Z_3 = Z_C = 1/jwC = 1/j(5)(20\times10^{-3}) = -10j\ \Omega$ -or- $10\angle -90°$

Substitute V and Z into the formula for Vout

$V_{out} = -25\angle 90° (10\angle -90° / 5\angle \mathrm{atan}(4/3)) - 6\sqrt{2}\angle 45° (10\angle -90° / 5\angle 0°)$

Multiply magnitudes and add phases, noting that a negative sign is worth 180°...

$V_{out} = (25*10/5)\angle[180° + 90° - 90° - \mathrm{atan}(4/3)] + (6\sqrt{2} * 10/6) \angle [180° + 45° - 90° - 0°]$

$V_{out} = 50 \angle [180° - \mathrm{atan}(4/3)] + 10\sqrt{2} \angle 135°$

Now we apply trigonometry to convert the phasors into rectangular form

If $\alpha = \mathrm{atan}(4/3)$, then $\cos(\alpha) = 3/5$ and $\sin(\alpha) = 4/5$ because $3^2 + 4^2 = 5^2$

Then $\cos(180° - \alpha) = -3/5$ and $j\sin(180° - \alpha) = +4j/5$

So... $50 \angle \{180° - \mathrm{atan}(4/3)\} = -30 + 40j$

$10\sqrt{2} \angle 135° = 10\sqrt{2} \cos(135°) + j10\sqrt{2} \sin(135°) = -10 + 10j$

$V_{out} = -30 + 40j - 10 + 10j = -40 + 50j$

Back to phasor form

$M = \mathrm{sqrt}(40^2 + 50^2) = 10\sqrt{(16 + 25)}$

$\theta = \mathrm{atan}(50/-40) = 180° - \mathrm{atan}(5/4) = 180° - 51.3° = 128.7°$

$V_{out} = 10\sqrt{41} \angle 128.7°$

Back to the time domain...

$v_{out} = 10\sqrt{41} \{\cos(5t + 128.7°)\}$ mV

1b. This circuit has four components whose values we might adjust. The current i_2 passes through a pure resistance so adjusting R_2 will not affect its phase. The capacitor affects the phase of both currents equally so it cannot be used to align the phases of i_1 and i_2. Therefore we want to select L and R_1 so that the phase of i_1 matches the phase of i_2.

$$\text{phase}(I_2) = \text{phase}(V_2) = 45°$$

$$\text{phase}(I_1) = \text{phase}(V_1/Z_1) = \text{phase}(V_1) - \text{phase}(Z_1) = 90° - \text{phase}(Z_1)$$

Therefore we want

$$\text{phase}(Z_1) = 90° - 45° = 45°$$

Where

$$Z_1 = R_1 + jwL$$

$$\text{phase}(Z_1) \equiv \arctan(wL/R_1) = 45°$$

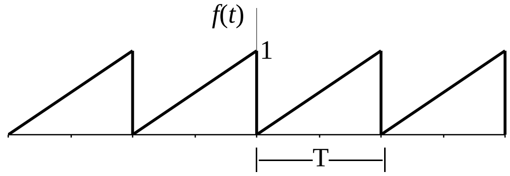
$$wL/R_1 = \tan(45°) = 1$$

Because w = 5 rad/sec, any change that makes $R_1/L = 5$ puts i_1 and i_2 in phase.

If we keep L = .8 H, then R1 = 4 Ω.

If we keep $R_1 = 3\ \Omega$, then L = .6 H.

2. (50) Let $f(t)$ be a sawtooth wave with period T. The function $f(t)$ can be represented by a Fourier series with complex coefficients, as shown below.



$$f(t) = C_0 + \sum_{\substack{-\infty \\ n\neq 0}}^{\infty} \boldsymbol{C}_n e^{jn\omega_0 t}$$

a) Find C_0 and an expression for the complex Fourier series coefficients $\boldsymbol{C}_n$.

2a. $f(t) = t/T$ with a period of T, so $\omega_0 = 2\pi/T$

$$C_0 = \frac{1}{T}\int_0^T t/T\,dt = \frac{1}{T^2}\left[t^2/2\right]_0^T = \frac{1}{T^2}\cdot\left(\frac{T^2}{2} - 0\right) = \frac{1}{2} = C_0$$

$$C_n = \frac{1}{T}\int_0^T \frac{t}{T} e^{-jn\omega_0 t}\,dt$$ Use integration by parts:

$$\int t e^{-\alpha t}dt = t\int e^{-\alpha t}dt - \int 1dt \int e^{-\alpha t}dt = \frac{t}{-\alpha}e^{-\alpha t} - \frac{1}{\alpha^2}e^{-\alpha t}$$

$$C_n = \frac{1}{T^2}\left[\frac{t}{-jn\omega_0}e^{-jn\omega_0 t} - \frac{1}{(jn\omega_0)^2}e^{-jn\omega_0 t}\right]_0^T$$

$$= \frac{1}{T^2}\left[\frac{T}{-jn\omega_0}e^{-jn\omega_0 T} - \frac{1}{(jn\omega_0)^2}e^{-jn\omega_0 T} - 0 + \frac{1}{(jn\omega_0)^2}e^{-jn\omega_0 T}\right]$$

Noting that $\omega_0 T = 2\pi$ and $e^{-jn2\pi} = 1$

$$C_n = \frac{1}{T^2}\left[\frac{T}{-jn\omega_0} - \frac{1}{(jn\omega_0)^2} + \frac{1}{(jn\omega_0)^2}\right] = \frac{1}{-jn\omega_0 T}$$

$$C_n = \frac{-1}{jn2\pi} \text{ OR } \frac{j}{n2\pi}$$

$$\therefore f(t) = \frac{1}{2} + \frac{j}{2\pi}\sum_{\substack{-\infty \\ n\neq 0}}^{\infty}\frac{1}{n}e^{+jn2\pi t/T}$$

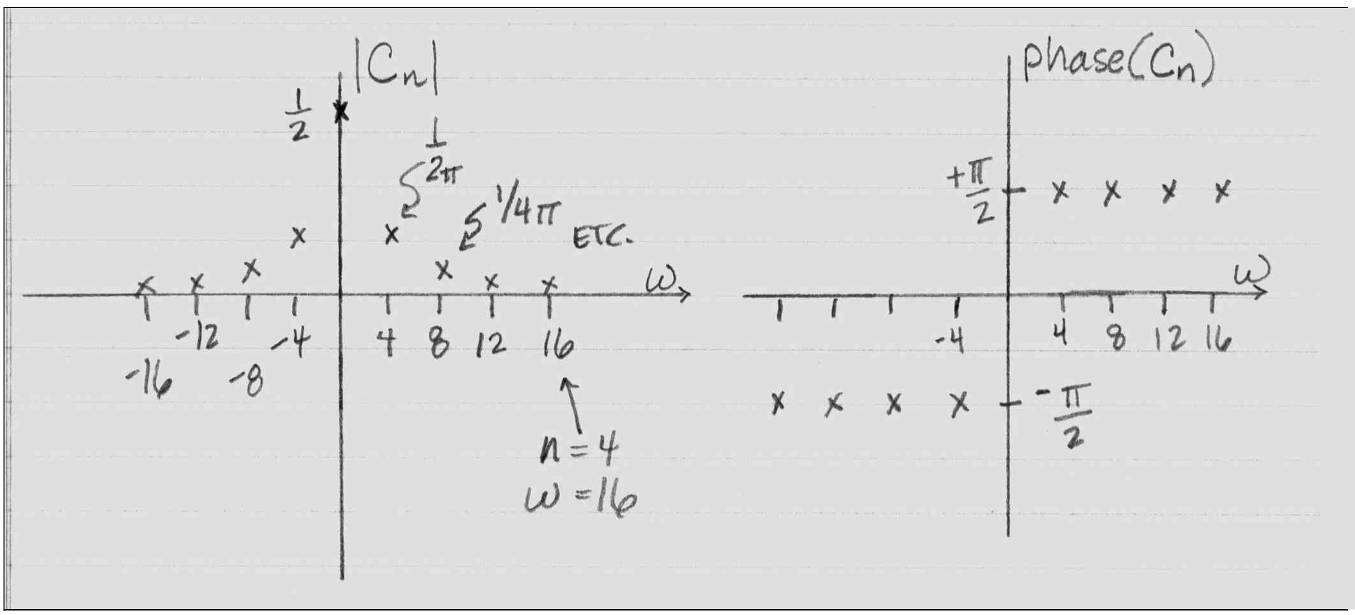
2b. $|C_0| = \frac{1}{2}$ $\qquad \angle C_0 = \operatorname{atan}(0) = 0°$

$$|C_n| = \left|\frac{J}{n2\pi}\right| \qquad \angle C_n = \operatorname{atan}\left(\frac{1/n2\pi}{0}\right)$$

$$= \left|\frac{1}{n2\pi}\right| \qquad \begin{cases} = \pi/2 \text{ or } 90° & \text{for } n>0 \\ = -\pi/2 \text{ or } -90° & \text{for } n<0 \end{cases}$$

When $T = \pi/2$, $\omega_0 = 4$

$2\omega_0 = 8$, etc.



2c. $$f(t) = \frac{1}{2} + \frac{j}{2\pi} \sum_{-\infty}^{\infty} \frac{1}{n} e^{+jn2\pi t/T}$$

$$= \frac{1}{2} + \frac{j}{2\pi} \sum_{-\infty}^{\infty} \frac{1}{n} \left[\cos(n2\pi t/T) + j\sin(n2\pi t/T)\right]$$

Because $\frac{1}{-n} = -\left(\frac{1}{n}\right)$ and $\cos(n2\pi t/T) = \cos(-n2\pi t/T)$,

$$\sum_{\substack{-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\cos(n2\pi t/T) = 0$$

Similarly,

$$\sum_{\substack{-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\sin(n2\pi t/T) = 2\sum_{1}^{\infty} \frac{1}{n}\sin(n2\pi t/T)$$

$$\therefore f(t) = \frac{1}{2} + 2\cdot\frac{j}{2\pi}\sum_{1}^{\infty}\frac{1}{n} j\sin(n2\pi t/T)$$

$$= \frac{1}{2} - \frac{1}{\pi}\sum_{1}^{\infty}\frac{1}{n}\sin(n2\pi t/T)$$

The solution may also be written using the period given in part 2b, such that the answer contains sin(4nt).