Lecture 12 Notes

November 8, 2010

System linearization

Linear systems can be modeled as one or more linear differential equations, i.e. equations that contain only linear operations. As we saw last week, linear operations (and systems) are those that satisfy superposition and proportionality criteria.

Each dependent variable that cannot change instantly (e.g. position and momentum or velocity) can be called a *state*. The order of the system or differential equation typically matches the number of states. The states may be derivates of other states, or may be separate values such as the voltages on capacitors that are separated by resistors.

The time-varying values of each state may be plotted as a function of time, or against each other in a phase portrait. Of course, it is difficult to draw phase portraits of systems that are higher than second order.

When a system is at equilibrium, none of the states are changing. That is, all of the derivatives are zero. Every linear system has only one equilibrium point, which is at the origin. In a second-order system that point is at (0,0), for example when position is zero and velocity is zero. This is true because all of the derivatives in the system are functions of velocity and position, so if velocity and position are zero, their derivatives must be too.

The path that a the system follows on a phase portrait from its initial conditions to (0,0) is called a trajectory. Trajectories can have many different shapes. The pattern of the trajectories around an equilibrium point is a useful indicator of the kind of system that created them, so each pattern is given a special name. For example, the phase portrait for a damped oscillatory system spirals inward from any given initial conditions, so it is called a spiral point. The solution to an undamped oscillatory system goes around and around a central point, so it is called a center. Other kinds of equilibrium points include nodes and saddle points.

Non-linear systems can have multiple equilibrium points. These multiple points mean that the overall phase portrait for a non-linear system can have patterns that are much more complicated than those for linear systems. Some are so complicated, in fact, that it is difficult to use them in a numerical model. Most importantly, transfer functions are all linear, so ***we cannot write a transfer function that represents a non-linear function.***

When the states in a non-linear system are close enough to an equilibrium point, the trajectories around that point look like the trajectories around (0,0) in a linear system. Therefore, we can pretend that the system is linear if a) we ignore the trajectories that are far from the equilibrium point, and b) we convert the non-linear differential equations into linear equations that are good approximations of the original non-linear equations. This process is called linearization.

To linearize a system of non-linear differential equations, we linearize the non-linear functions in these differential equations. For trig functions, linearization is usually called the small-angle approximation. For example, sin(α) ≈ α when α is near 0 or 2π*n*, where *n* is an integer.

For functions that are not trigonometric, we use a more general formula, which gives us a first-order approximation of *f*(*x*) about the point *x*=*a*:

$f(x)\big|_{x\approx a} \approx f(a) + \frac{df}{dx}\bigg|_{x\approx a} x$. These are the first two terms in a Taylor series. When we linearize a system we ignore the other terms with higher order derivatives – not because of the derivatives, but because they are multiplied by x^2, x^3, etc.

As you saw in lecture (and probably PHYS 121), the differential equation for an unforced, undamped pendulum is $\ddot{\theta}(t) + \frac{g}{L}\sin\theta(t) = 0$. Applying the small-angle approximation linearizes the equation to $\ddot{\theta}(t) + \frac{g}{L}\theta(t) = 0$. We could also say that we are linearizing the system about $(\theta, \dot{\theta}) = (0,0)$. Omitting many steps, and assuming that the initial position is 1 and the initial velocity is zero, the solution to this equation is an undamped sinusoid, $\theta(t) = \cos\sqrt{\frac{g}{L}}t$. This is a *simple harmonic oscillator*.

We can also linearize the differential equation around another angle, for example $\theta=\pi/2$. This angle is not an equilibrium point because $d^2\theta/dt^2 = -g\sin(\theta)/L$ and sin(π/2) = 1. Therefore, the rate(s) of change there are not zero.

Important point: We can linearize a *function*, or a differential equation, about any point we want. However, we can linearize a *system* – that is, approximate the system as linear – only around an equilibrium point. This is true because all linear systems have only one equilibrium point – the one at (0,0). If there is not an equilibrium point at its center, it's not a linear system.

To linearize the pendulum equation (not the system) where the pendulum is straight sideways, we use the linear (small angle) approximation around θ = π/2, noting that the derivative of sine is cosine.

$$\sin(\theta) \approx \sin(\pi/2) + \cos(\pi/2)(\theta - \pi/2) = 1.$$

Therefore, the differential equation becomes approximately

$$\ddot{\theta}(t) + \frac{g}{L} = 0, \text{ or } \ddot{\theta}(t) = -\frac{g}{L}.$$

Integrating twice gives $\theta(t)=\frac{g}{2L}t^2+C_1t+C_2$, or, using the initial conditions,

$\theta(t)=\frac{g}{2L}t^2+\dot{\theta}(0)t+\theta(0)$, which produces a parabolic shape in the phase portrait.

<u>For next time</u>

There is an equilibrium point at $\theta=\pi$, so we can linearize the system there.

From before, $\ddot{\theta}(t)+\frac{g}{L}\sin\theta(t)=0$

Approximate sin(θ) when θ is near π:

$$\sin(\theta)\approx\sin(\pi)+\cos(\pi)(\theta-\pi)=-(\theta-\pi)$$

We are trying to linearize the system, i.e. to put an equilibrium point at (0,0). To do this we need to shift the origin of our coordinate system by defining a new variable, $q=\theta-\pi$. Note that the derivatives of q and θ are equal because derivation eliminates the constant π. Therefore, the resulting linear differential equation is:

$$\ddot{q}(t)-\frac{g}{L}q(t)=0$$

The characteristic equation is $r^2-g/L=0$, and the roots are $r=\pm\ g/L$. The resulting solution is

$$q(t)=Ae^{gt/L}+Be^{-gt/L}$$

$$A=\frac{3}{2}q(0)-\frac{g}{2L}\dot{q}(0)$$

$$B=\frac{1}{2}\left(\frac{g}{L}\dot{q}(0)-q(0)\right)$$

I am not sure about the formulas for A and B; you are welcome to submit corrections.

The solution has an positive exponent, showing that the pendulum falls away from the equilibrium point if it is displaced from its exact equilibrium position. It is theoretically possible to give the pendulum the precise initial velocity and position such that it rises to the equilibrium position and stays there, but in almost every case it would come close and then fall away again. The resulting pattern of trajectories near this equilibrium point gives it the name *saddle point*. The math 307 book has some nice pictures of saddle points. Wikipedia shows some 3-D saddle points but not a 2-D picture.

The MATH 307 book (6th edition, at least) also shows a phase portrait for the whole non-linear pendulum system.

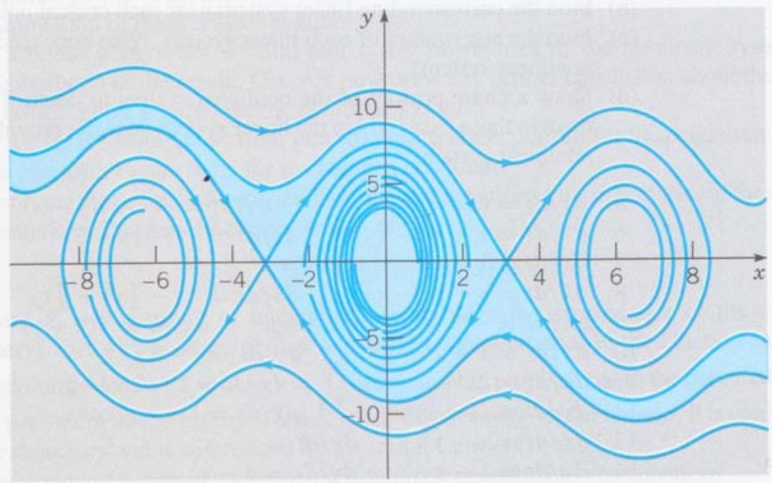

FIGURE 9.3.5 Phase portrait for the damped pendulum of Example 4.

Note that the horizontal axis is in radians and the vertical axis must be in radians/second. The actual amplitude depends on the particular length of pendulum that they used.

You can see the spiral point at (0,0) where the lines spiral inward, the almost-parabolic section near (1.57,0), the saddle point at (3.14,0), and the repetition of the spiral point at (±6.28,0), which is of course physically the same as (0,0).

Coming soon: what to do with two-state systems where one state is *not* the derivative of the other.