BIOEN 302

Lecture 16

Fourier Transforms

November 29, 2010

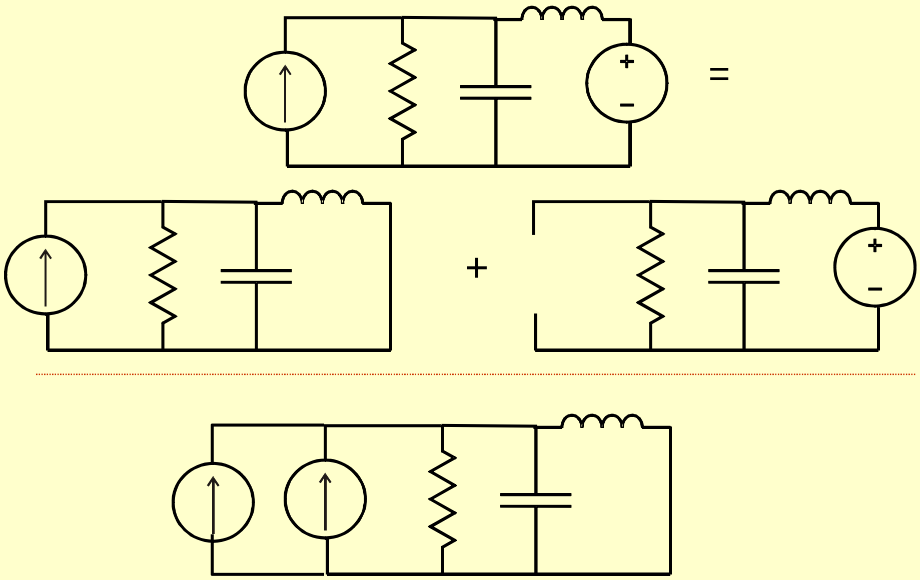
Fourier Series: main points

- Infinite sum of sines, cosines, or both

$$a_0 + \sum_{1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

- All frequencies are integer multiples of a fundamental frequency, ω_o
- F.S. can represent any periodic function that we can physically produce

Underlying principle: superposition



Fourier coefficients: trig form

$$f(t) = a_0 + \sum_{1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{1}{T} \int_{0}^{t0+T} f(t)\,dt$$

$$a_k = \frac{2}{T} \int_{0}^{t0+T} f(t) \cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T} \int_{0}^{t0+T} f(t) \sin(k\omega_0 t)\,dt$$

Source of the Fourier coefficients

$$\int_0^{t0+T} \cos(m\omega_0 t)\sin(n\omega_0 t)dt = 0, \text{ all } m \text{ and } n$$

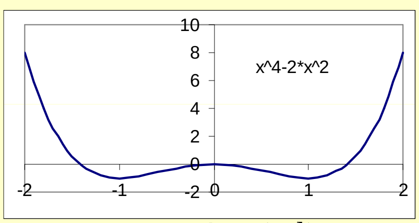
$$\int_0^{t0+T} \cos(m\omega_0 t)\cos(n\omega_0 t)dt = 0, \; m \neq n$$

$$\int_0^{t0+T} \sin(m\omega_0 t)\sin(n\omega_0 t)dt = 0, \; m \neq n$$

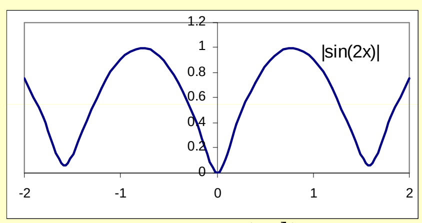
$$\int_0^{t0+T} \cos^2(m\omega_0 t)dt = \int_0^{t0+T} \sin^2(m\omega_0 t)dt = \frac{T}{2}$$

Symmetry of functions

- Even symmetry: *f*(*t*) = *f*(–*t*)

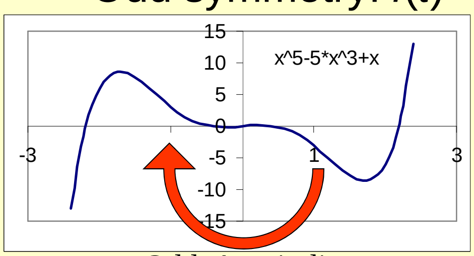


Even, Aperiodic

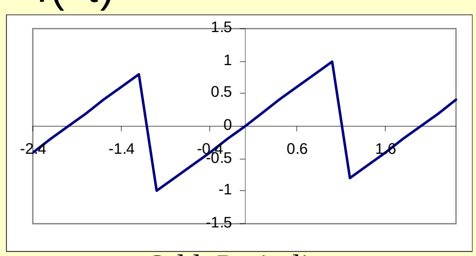


Even, Periodic

- Odd symmetry: *f*(*t*) = –*f*(–*t*)



Odd, Aperiodic



Odd, Periodic

Fourier coefficients: Complex exp. form

- $$f(t) = \sum_{-\infty}^{\infty} C_n e^{jn\omega_0 t}$$
  where
  $$C_n = \frac{1}{T} \int_{t0}^{t0+T} f(t) e^{-jn\omega_0 t}$$
- Example: for periodic pulse train

$$C_n = \frac{V_m \tau}{T} \sum_{-\infty}^{\infty} \frac{\sin(n\omega_0 \tau / 2)}{n\omega_0 \tau / 2}$$

$$\frac{\sin(x)}{x} = \mathrm{sinc}(x)$$

Magnitude and phase plots

- Magnitude plot shows $|C_n(\omega)|$
- Phase shows $\tan^{-1}(\mathrm{Im}\{C_n\}/\mathrm{Re}\{C_n\})$
- Plot exists at $n\omega_0$ only

Magnitude and phase plot - example

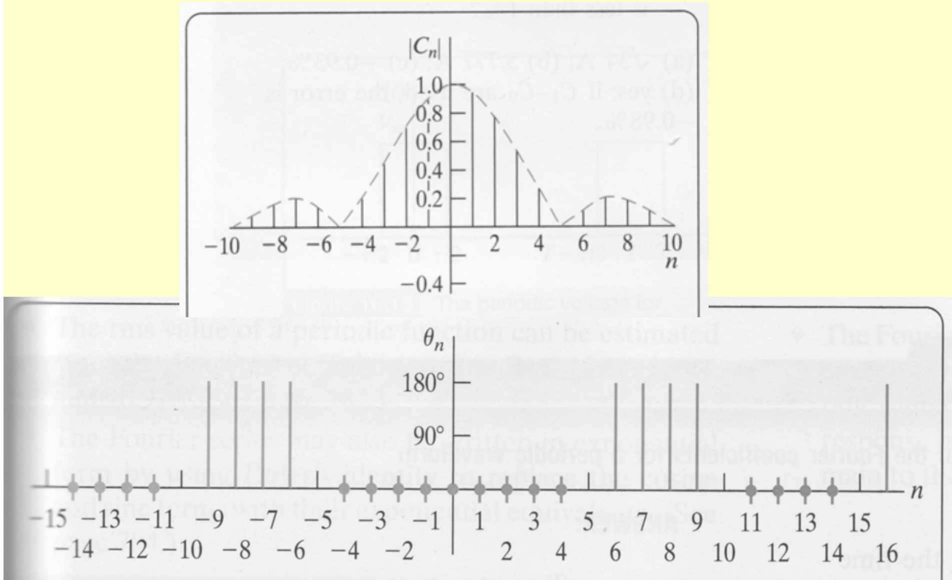


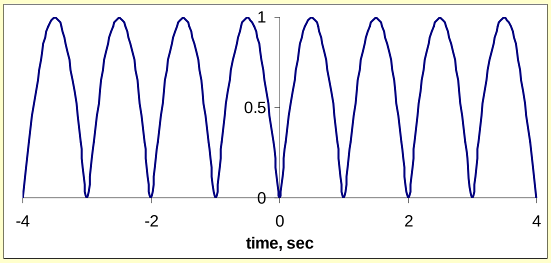
Figure 16.18 The phase angle of C_n.

Fourier Series: scaling property

$$f(t) = \sum_{-\infty}^{\infty} C_n e^{jn\omega_0 t}, \quad C_n = \frac{1}{T}\int_{t0}^{t0+T} f(t) e^{-jn\omega_0 t}$$

- Complex coefficients:
  - Magnitude and ω_0 vary inversely with *T*
  - Coefficients become smaller and more closely spaced as period increases

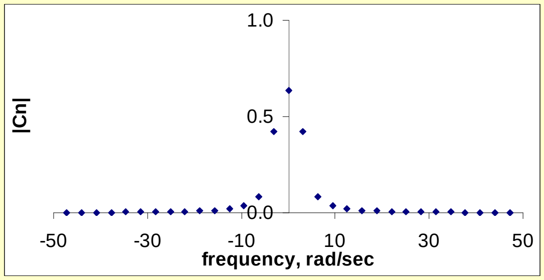
Fourier Series: scaling property



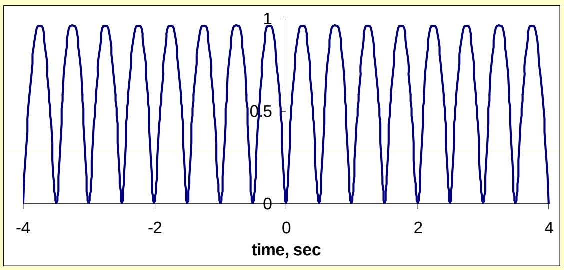
Full-wave rectified sine

Time domain

Frequency domain



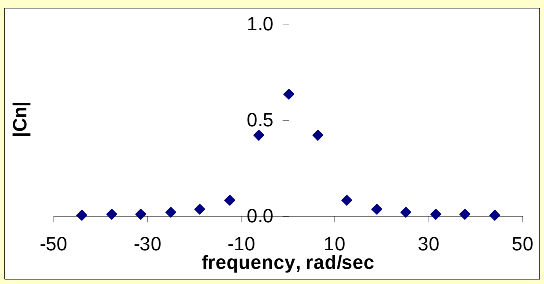
Fourier Series: scaling property



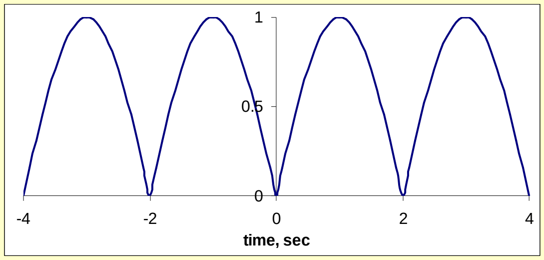
Full-wave rectified sine, $T_2=T_1/2$

Time domain

Frequency domain



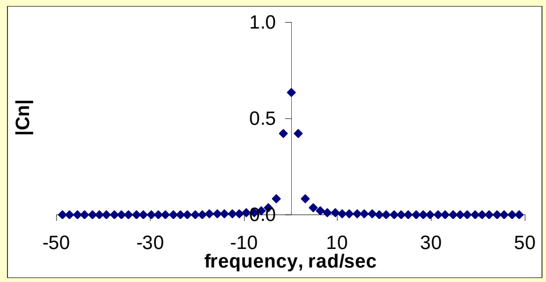
Fourier Series: scaling property



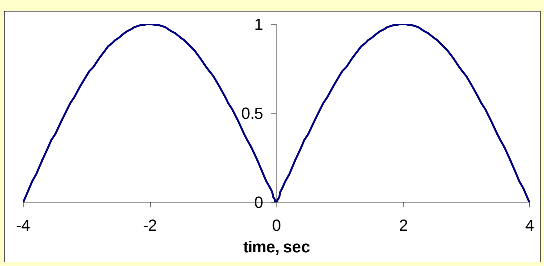
Full-wave rectified sine, $T_2 = 2T_1$

Time domain

Frequency domain



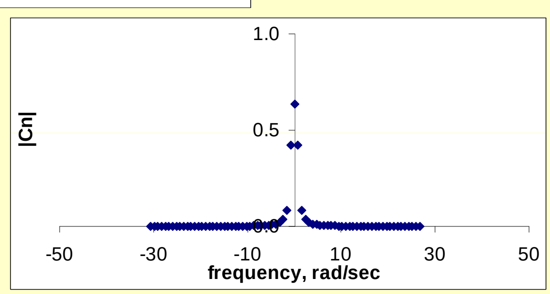
Fourier Series: scaling property



Full-wave rectified sine, $T_2 = 4T_1$

Time domain

Frequency domain



and so on....

Fourier Series: scaling

- Things to note:
  - As $T \to \infty$, $\omega_0 \to 0$, i.e. the series becomes continuous
  - As $T \to \infty$, $|C_n| \to 0$ but the sum of the coefficients over an interval ω_1 to ω_2 remains finite
  - Therefore, we can calculate $C_n T$, which is a finite, continuous function of ω
  - The resulting function is...

The Fourier Transform

Radian form

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega$$

Hertz form

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt$$

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df$$

The Fourier Transform

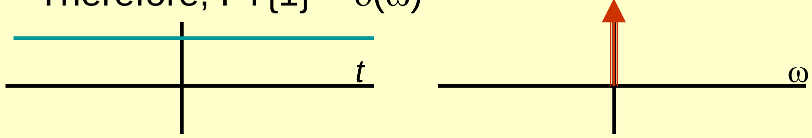
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

- Things to note:
  - The FT is a weighting function for sinusoidal (or complex exponential) content in signal
  - The FT transforms a continuous, aperiodic function in time...into an aperiodic, continuous function in frequency
  - Because both the FT and IFT contain complex exponentials, there are many cases of duality among FT pairs

A few Fourier pairs

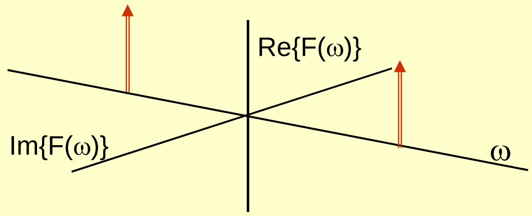
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

- $f(t) = 1$ (or any constant)
  - This function has zero frequency (or infinite period)
  - The area under the FT curve must be finite for the amplitude of the time-domain signal to be non-zero
  - Therefore, FT{1} = $\delta(\omega)$



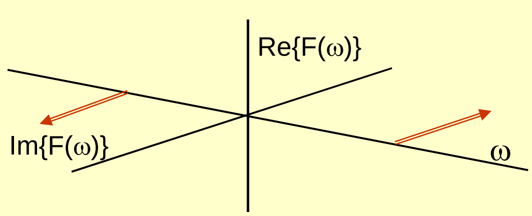
A few Fourier pairs

- *f*(*t*) = cos(ω_0t)
  - This function has a specific frequency
  - Negative and positive frequencies are both present to cancel the imaginary part
  - Therefore,
    $\mathcal{F}\{\cos(\omega_0 t)\} = (2\pi/2)[\delta(\omega+\omega_0)+ \delta(\omega-\omega_0)]$
  - Note that $\mathcal{F}\{\cos(\omega_0 t)\}$ is real and even



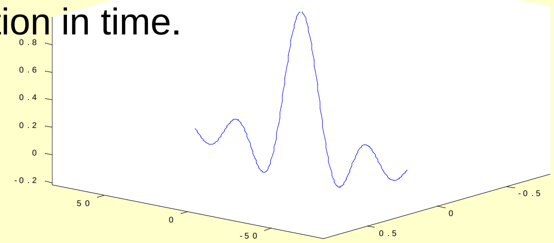
A few Fourier pairs

- *f*(*t*) = sin(ω_0t)
  - This function has a specific frequency
  - A 90° phase shift in a complex exponential means multiplication by *j*
  - Therefore,
    $\mathcal{F}\{\sin(\omega_0 t)\} = j\,\pi\,[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$
  - Note that $\mathcal{F}\{\sin(\omega_0 t)\}$ is imaginary and odd



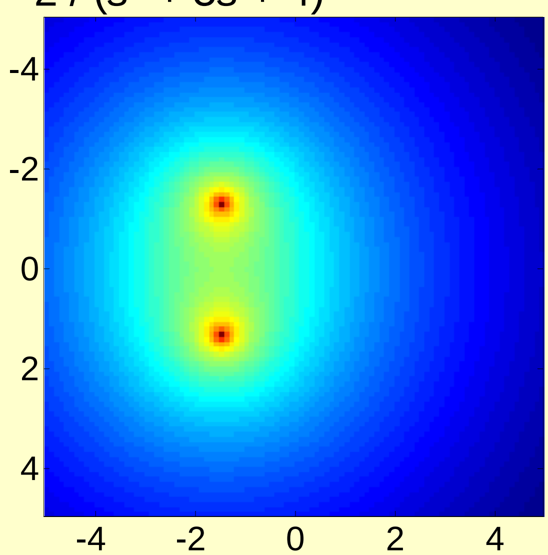
f(*t*) = square pulse of height *A* and width 2*b*, *centered at t = 0*

- $F(\omega) = 2Ab \sin(\omega b)/\omega b = 2Ab\ \text{sinc}(\omega b)$
  - The amplitude $F(0)$ is the area under the pulse
  - $F(\omega)$ is real and even if the pulse is centered on $t=0$.
  - $F(\omega)$ is complex if the pulse is not centered.
  - The IFT of a pulse in frequency is a sinc function in time.



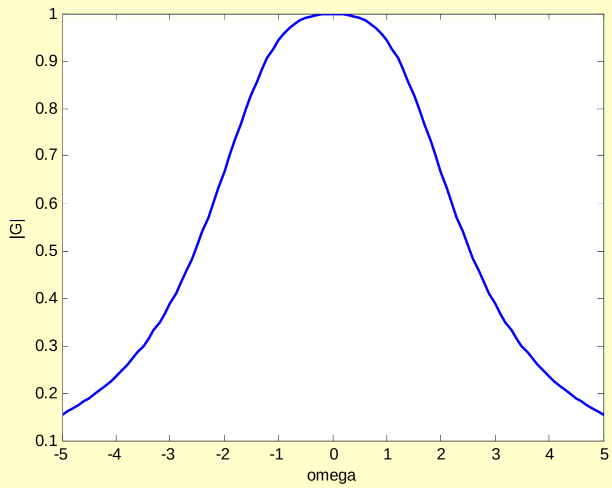
Transfer functions in the s-plane

$H(s) = 2 / (s^2 + 3s + 4)$



Magnitude plot

- $G(\omega)$ = 4/((jw)^2 + 3(jw) + 4)



For image analysis:
Two-dimensional Fourier transformation

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{+\infty} f(x,y) e^{j2\pi(xu+yv)} dxdy$$

- Integral form
  - Operates on continuous image
  - Does not assume image is periodic

For image processing:
Two-dimensional Fourier transformation

$$F(m,n) = \sum_{x=1}^{M} \sum_{y=1}^{N} f(x,y) e^{j2\pi(xm+yn)}$$

- Summation form
  - Usually called “Discrete Fourier Transform”
  - Operates on sampled image (not continuous)
  - Assumes the image is periodic in *x* and *y*

Operational transforms

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

- Translation in the time domain
  - Equivalent to multiplication in the frequency domain
- Translation in the frequency domain
  - Equivalent to multiplication by complex exponential in the time domain
  - *Not a smart thing to do*
- $F(\omega)$ is real and even if the pulse is centered on t=0.

Operational transforms

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

- Scale change
  - When one domain is stretched out, the other domain is compressed
  - Example: T increases, ω_0 decreases
  - Wider in time means narrower *and taller* in frequency

Operational transforms

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

- Modulation (e.g. AM radio)
  - Amplitude of high-frequency carrier is modified by amplitude of low-frequency signal
  - $\mathcal{F}\{f(t)\cos(\omega_0 t)\} = \frac{1}{2} F(\omega+\omega_0) + \frac{1}{2} F(\omega-\omega_0)$
  - The original signal would have $|F(\omega)|$ centered around $\omega=0$; the modulated signal would have $|F(\omega)|$ duplicated and shifted along the ω axis.
  - Interesting... What was $\mathcal{F}\{\cos(\omega_0 t)\}$?

Operational transforms

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

- Convolution
  - The output *y*(*t*) from a system with unit impulse response *h*(*t*), when the input is *x*(*t*), can be represented in two ways:
    - by convolution in the time domain
    - by multiplication in the frequency domain
      $Y(\omega) = X(\omega)H(\omega)$
      Similar to $V_{OUT}(s) = V_{IN}(s)H(s)$ in Laplace domain

Laplace vs. Fourier

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

- Laplace transforms are better
  - For systems analysis (convergence for wider variety of functions)
  - For control systems analysis
- Fourier transforms are better
  - Easier to understand $j\omega$ axis than *s* plane
  - Basis for FFT for discrete data
  - Widely used in signal processing