LTI System and Control Theory

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Contents

Chapter 1

LTI Mathematical Fundamentals

In this chapter we will continue to analyze dynamic systems; however we will be looking at systems in a context that lends itself to the description of physical systems in the Laplace domain. In previous sections we have simply taken a system and observed the system's Fourier (i.e., frequency domain) transfer function. Although the frequency spectrum produced by a system elucidates much of the behavior of a system, it does not lend itself to physical modeling as it ignores any internal states within the system. Using Laplace domain techniques we may better understand the physical processes driving a system's behavior and thereby create more accurate system models.

This chapter provides an introduction to the analysis of single input single output linear dynamical systems from a mathematical perspective, starting from the simple definitions and assumptions required by linear time-invariant (LTI) systems and continuing through the study of LTI system transfer functions and analysis methods. The techniques described in this chapter will be used extensively in the next chapters which describe active filter design and feedback control system design.

1.1 Definitions and Representation

The hallmark of linear time-invariant systems is their time varying nature that can be modeled deterministically using differential equations. Most physical systems fall into this category.

1.1.1 Linearity and Time Invariance

For a system to be considered an LTI system it must exhibit two properties, linearity and time invariance. These two properties are defined below.

Linearity

To understand the property of linearity, it is often useful to recall the basic definition of a line. A line is often defined as:

$$y = l(x) = mx + b \tag{1.1}$$

where x and y are the input and output variables, respectively, m is the slope parameter and b is the intercept parameter. Notice that the output variable is dependent on two constants, a scaling factor and a translation factor (m and b, respectively). From this we may infer that the crucial factors in defining a linear system are scaling and superposition. Thus by definition, a linear system represented as $y = f(x)$ will have the following properties:

$$\begin{aligned} f(\alpha x) &= \alpha f(x) \text{ Scaling} \\ f(x_1 + x_2) &= f(x_1) + f(x_2) \text{ Superposition,} \end{aligned}$$

where α is a constant. Linearity is a crucial concept for LTI system theory because it allows us to use various linear operations and transformations on our system so that we may better understand it or manipulate it. Although most realistic systems are nonlinear, we are often able to assume that a system is linear or nearly linear within a certain range (e.g., the small angle approximation of the function $\sin(x)$). This will be discussed further in section 1.1.4.

Example 1 Is the function in equation 1.1 linear?

Solution. To prove that the function $l(x)$ in equation 1.1 is linear, we must test it using the properties described above. First, testing the scaling property, we find that:

$$l(\alpha x) = m\alpha x + b \neq \alpha(mx + b). \tag{1.2}$$

Therefore $l(x)$ is, interestingly, not linear if b is non-zero. However, if we set b to be zero, then

$$l(\alpha x) = m\alpha x = \alpha(mx), \tag{1.3}$$

making $l(x)$ satisfy the scaling property of linearity.

Now does $l(x)$ such that $b = 0$ also satisfy the superposition property? Using the previous definition,

$$l(x_1 + x_2) = m(x_1 + x_2) = mx_1 + mx_2 = l(x_1) + l(x_2), \tag{1.4}$$

where x_1 and x_2 are arbitrary input variables. This satisfies the superposition property. Therefore, we have shown that $l(x)$ is not linear if $b \neq 0$ but is linear if $b = 0$.

Time invariance

Time invariance, also known as shift invariance, describes a function's independence from the location of $t = 0$ on the time line. By definition, a time-invariant system's output will shift in time if its input shifts in time, but otherwise will remain exactly the same. In other words, a time-invariant function does not care what time it is. We describe time invariance with the following notation:

$$\text{Given:} \qquad y(t) = f(t), \tag{1.5}$$
$$\text{then:} \qquad y(t - \delta) = f(t - \delta) \tag{1.6}$$

Shift invariance is a crucial property of real systems because it allows us to assume that a system will respond in a predictable manner at any time. Modeling time dependent systems are often highly influenced by initial conditions and system definitions.

Example 2 Is the system $y = t + x(t)$ time invariant?

Solution. To prove whether or not the above system is time-invariant we must a mathematical technique called proof by contradiction. Proof by contradiction is often used when two separate conditions can be tested on the same system or mathematical construct. In this case, we can compare the result of the time shifted system with the solution assuming that the system is time-invariant. Thus, the shifted system is represented as:

$$y_1(t) = y(t - \delta) = (t - \delta) + x(t - \delta). \tag{1.7}$$

Now, if we redefine the input function to be $x(t-\delta) = x_{shift}(t)$ which we may assume if and only if the system is time invariant, then we represent the original system as:

$$y_2(t) = t + x_{shift}(t) = t + x(t - \delta) \tag{1.8}$$

Since $y_1 \neq y_2$, we may conclude that the system is not time invariant through proof by contradiction. In other words, if $y(t)$ contains a t outside of the input function $x(t)$ it is time variant.

Example 3 Is the system $y = \alpha x(t) + \beta$ time invariant? (note: α and β are constants)

Solution. We will use the same method again on the new system described above. The shifted system is represented as:

$$y_1(t) = y(t - \delta) = \alpha x(t - \delta) + \beta. \tag{1.9}$$

Again, if we let $x(t-\delta) = x_{shift}(t)$ (and thereby assume the system is time invariant), then we represent the original system as:

$$y_2(t) = \alpha x_{shift}(t) + \beta = \alpha x(t - \delta) + \beta \tag{1.10}$$

Since $y_1 = y_2$, we may conclude that the system is time-invariant.

1.1.2 Linear Sets of Differential Equations

By now the reader should be well acquainted with ordinary differential equations (if not, please refer to the BIOEN 302 mathematics review, Chapters 7 and 8 of the EE 215 textbook, or to your MATH 307 book). When an ODE has only one first-order derivative, there is no need to break it into simpler equations. However, linear ODEs with higher order derivatives can be rewritten as multiple first-order equations where each equation represents the behavior of one system state. These equations can then be solved simultaneously using advanced methods or can be converted to a transfer function using a modification of the traditional Laplace transformation.

Examples of system states are the voltage across the capacitor in an RC circuit, the momentum of water moving through a pipe, or the amount of a drug in the bloodstream. The defining feature of a system state is that it cannot change instantaneously. Therefore, system states are typically the source of our initial conditions.

We will explore this new notation through two examples, the first being the conversion of a familiar second order ODE into a system of two differential equations. The second example is a higher order system that would be more difficult to represent as a single differential equation. In both cases we have one input and one output, so we refer to them as Single-Input, Single-Output (SISO) systems.

Example 4 Representing an LRC circuit as a system of ODEs.

Solution. A series LRC circuit is shown in figure 1.1. To write out the standard form ODE for this circuit, the best method is to use Kirchhoff's voltage law,

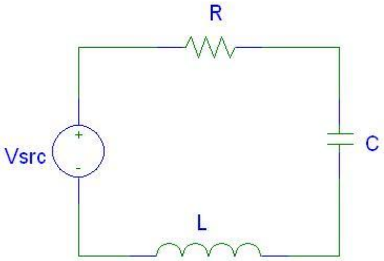


Figure 1.1: Circuit schematic of a series connected LRC circuit.

which states that the sum of the voltages around a complete loop must equal zero. Symbolically this is represented as:

$$0 = V_L + V_R + V_C - V_{src}. \tag{1.11}$$

The equation can be simplified by knowing that the current $i(t)$ is uniform throughout the circuit. Thus, the differential equation can be rewritten in terms of this current:

$$V_{src} = L\frac{di}{dt} + iR + \frac{1}{C}\int idt \tag{1.12}$$

or

$$\frac{1}{L}V_{src} = \frac{di}{dt} + \frac{R}{L}i + \frac{1}{LC}\int idt. \tag{1.13}$$

Now that we have established the original ODE we can convert it to a linear system of differential equations by using a simple change in notation, where we will create two states. The first state, x_1, will represent the charge on the capacitor, while x_2 will represent the current flowing through the circuit. Because charge is the integral of current over time,

$$x_1 = \int idt = \int x_2 dt \tag{1.14}$$

and

$$x_2 = \dot{x_1} \equiv dx_1/dt \tag{1.15}$$

Using these definitions, we may rewrite equation 1.13 as a set of two dependent differential equations:

$$\dot{x_1} = x_2 \tag{1.16}$$

$$\dot{x_2} = \frac{1}{LC}x_1 + \frac{R}{L}x_2 - \frac{1}{L}V_{src}. \tag{1.17}$$

Since we now have two states in the system, it is no longer explicit which state is the output of our linear set of ODEs. Thus, since we have defined current, our parameter of interest, as x_2, we must explicitly define the output of our system as:

$$y = 0x_1 + 1x_2 = x_2. \tag{1.18}$$

It is important to note that the order of the differential equation is equivalent to the number of states necessary to represent the system dynamics.

$\prec == \succ$

Although it may seem trivial to convert an nth order ODE to a system of n 1st order ODEs may seem trivial, there are a large number of analytical tools that can use only this first-order form of differential equations. In addition, some systems can be written as a set of linear ODEs much more easily, as shown in the following example.

Example 5 Representing a chemical reaction network as a system of ODEs.

Solution. The chemical reaction network to be analyzed is shown in figure 1.2. This system could be a metabolic pathway used to manufacture chemical $S4$ from $S1$. $S2$ and $S3$ would then be reaction intermediates for this process. To describe the dynamics of this system, we may utilize first-order mass-action kinetics. As one may recall from general chemistry, mass action kinetics describes the rate of production or degradation of a species as the sum of the mass flux through that species (similar to Kirchhoff's current law).

Applying mass-action kinetics, we can directly write the differential equations which describe the change in the species concentration for each species. For example,

$$\dot{S}1 = u - (k_1S_1 + k_3S_1), \tag{1.19}$$

where $\dot{S}1$ is the time derivative of the concentration of species $S1$, u is the system input and is the only influx to the state $S1$, and $(k_1S_1 + k_3S_1)$ is the gross outflux

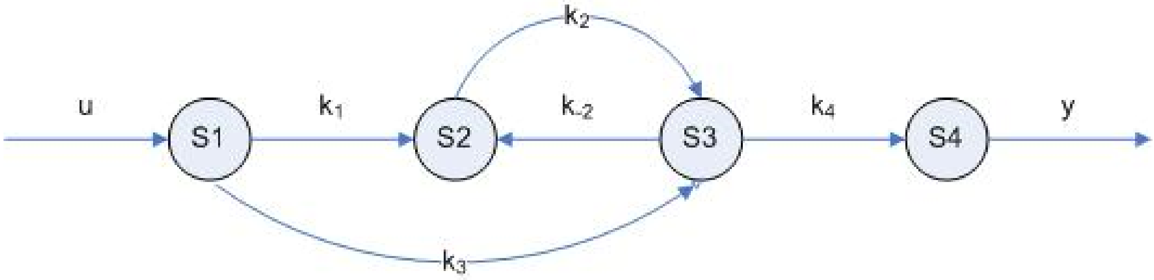


Figure 1.2: Example of a chemical network with four intermediate species, labeled S1 through S4.

from $S1$. Using the same reasoning we can write the differential equations for all of the system states:

$$\dot{S}1 = u - k_1 S_1 - k_3 S_1 \tag{1.20}$$

$$\dot{S}2 = k_1 S_1 + k_{-2} S_3 - k_2 S_2 \tag{1.21}$$

$$\dot{S}3 = k_2 S_2 + k_3 S_1 - k_{-2} S_3 - k_4 S_3 \tag{1.22}$$

$$\dot{S}4 = k_4 S_3 \tag{1.23}$$

$$\text{and } y = S_4 \tag{1.24}$$

This example shows how writing ODEs in this form can be much more intuitive than trying to represent the entire system as a single differential equation.

1.1.3 The State Space Representation of Linear Differential Equations

Although a system of first-order differential equations lends itself to describing some types of systems better, it is still difficult to do any math on a linear system of differential equations in its raw form. Therefore, a formalized method for representing linear systems of differential equations has been developed called the state space formalization, which utilizes matrix notation to represent linear sets of ODEs in a mathematically relevant way. The state space formalization will be described in this section.

Recall from linear algebra that the following linear system:

$$y_1 = 2x_1 + x_2 \tag{1.25}$$

$$y_2 = x_1 - x_2 \tag{1.26}$$

may also be represented as:

$$\vec{y} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \vec{x} \tag{1.27}$$

where $\vec{x} = [x_1 \; x_2]^T$ and $\vec{y} = [y_1 \; y_2]^T$. Notice that when you multiply $\vec{x}$ with the matrix, you will get a 2x1 matrix equal to equations 1.25 and 1.26.

In example 4 we showed how to write a second-order differential equation as a set of two first-order equations by defining charge as x_1 and current as x_2. To put the LRC system into matrix form, we would write the equations

$$\dot{x_1} = 0x_1 + 1x_2 + 0V_{src} \tag{1.28}$$

$$\dot{x_2} = \frac{1}{LC}x_1 + \frac{R}{L}x_2 - \frac{1}{L}V_{src} \tag{1.29}$$

then factor out the all of the state terms to get

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 1/LC & R/L \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ -1/L \end{bmatrix} V_{src}, \tag{1.30}$$

where $\vec{x}$ is the state vector (a 2x1 column vector containing the two system states (x_1 and x_2), and $\dot{\vec{x}}$ is the time derivative of the state vector (i.e., $\dot{\vec{x}} = [\dot{x_1}\ \dot{x_2}]^T$). We often call the 2x2 matrix of constants the state matrix and the 2x1 matrix of constants multiplying the input scalar (V_{src} in this case) the input matrix. We are, however, not done. Since we are now using multiple states to describe this system we must also specify the output explicitly. Previously we wrote the output of the LRC system as

$$y = 0x_1 + 1x_2. \tag{1.31}$$

because we wanted to know the current. We may convert this to matrix notation as

$$y = [0\ 1]\vec{x}, \tag{1.32}$$

where $[0\,1]$ is often described as the output matrix. If the output is one of the states, then the output matrix will contain one 1 and the rest zeroes. If we wanted to know the voltage across the capacitor, then the output vector would be

$$y = [1/C\ 0]\vec{x}, \tag{1.33}$$

where C is the capacitance.

Each matrix described above is often designated with a specific letter which is standard notation for state space form. These are: A for the state matrix, B for the input matrix, and C for the output matrix. Thus, we may define the state space formalization as

$$\begin{aligned} \dot{\vec{x}} &= \mathrm{A}\vec{x} + \mathrm{B}u \qquad (1.34) \\ y &= \mathrm{C}\vec{x} + \mathrm{D}u, \qquad (1.35) \end{aligned}$$

where $\vec{x}$ is the state vector and u is the system input. Later we will use $u(t)$ to denote the unit step function. This is not merely a coincidence, because a step change is the most common system input. For now, however, the input u above is meant to be any function, not just a step. Note that in this form there is also a matrix D which describes how the input function might directly affect the output of the system. For most SISO systems the value of D will be zero. Therefore, for the LRC system we would define our four matrices as:

$$A = \begin{bmatrix} 0 & 1 \\ 1/LC & R/L \end{bmatrix} \tag{1.36}$$

$$\mathrm{B} = \begin{bmatrix} 0 \\ -1/L \end{bmatrix} \tag{1.37}$$

$$\mathrm{C} = [0\;1] \tag{1.38}$$

$$\mathrm{D} = 0 \tag{1.39}$$

Some things to note:

1. A is always a square nxn matrix where n is the number of system states;

2. B is always a nx1 matrix where n is the number of system states if the system is SISO;

3. C is always a 1xn matrix where n is the number of system states if the system is SISO;

4. D is always a scalar if the system is SISO.

Example 6 Rewrite the set of linear differential equations in example 5 in state space form.

Solution. Given the differential equations:

$$\begin{aligned} \dot{S1} &= u - k_1S_1 - k_3S_1 && (1.40)\\ \dot{S2} &= k_1S_1 + k_{-2}S_3 - k_2S_2 && (1.41)\\ \dot{S3} &= k_2S_2 + k_3S_1 - k_{-2}S_3 - k_4S_3 && (1.42)\\ \dot{S4} &= k_4S_3 && (1.43)\\ \text{and } y &= S_4 && (1.44) \end{aligned}$$

we would rewrite this system in state space form as:

$$\dot{\vec{S}} = \begin{bmatrix} -(k_1+k_3) & 0 & 0 & 0 \\ k_1 & -k_2 & k_{-2} & 0 \\ k_3 & k_2 & -(k_{-2}+k_4) & 0 \\ 0 & 0 & k_4 & 0 \end{bmatrix} \vec{S} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \tag{1.45}$$

$$y = [0\;0\;0\;1]\vec{S}, \tag{1.46}$$

where $\vec{S} = [S1\; S2\; S3\; S4]^T$.

1.1.4 Sets of Nonlinear Differential Equations

In the previous sections we have defined linearity and discussed various representations of linear systems of ODEs. However most real systems are nonlinear. Why then, you might ask, have we exerted all of this time to understand linear systems. As it turns out there are many systems that, although nonlinear, exhibit properties very similar to their linear counterparts. In addition, there are various techniques which will allow us to use linear analysis on nonlinear systems. One powerful technique in nonlinear analysis is the linearization of nonlinear systems, which will be described below.

Equilibrium Points

One of the most fundamental and important pieces of information that can be obtained from a set of ODEs is the system's natural equilibrium points. An equilibrium point is a point in the state space where the time derivatives of *all* the states are equal to zero. In any stable system, the natural response will tend toward equilibrium as $t \to \infty$ There is no set way to solve for the equilibrium points, but there are a variety of techniques that can be used to find the equilibria in dynamical systems. (Note: the techniques described in this section are analytical and there are other graphical techniques described in section 2.1.2 which can also be used to find equilibrium points a system's state space).

Equilibria of Linear Systems First, we will start with the trivial case of finding the equilibrium point for a linear system of differential equations. Since we are dealing with the natural response of the system, we are able to set u to zero (note that this analysis can also be conducted with constant values of the input variable, u). For the generic first order system:

$$\dot{x_1} = a_{(1,1)}x_1 + a_{(1,2)}x_2 \tag{1.47}$$

$$\dot{x_2} = a_{(2,1)}x_1 + a_{(2,2)}x_2 \tag{1.48}$$

we may solve for the equilibrium point(s) of this system by converting the above differential equations into state-space form and solving for $\vec{x}$ when $\dot{\vec{x}}$ is set to zero (i.e., the definition of equilibrium). Thus, we may rewrite our equations as:

$$\dot{\vec{x}} = 0 = \mathrm{A}\vec{x} = \begin{bmatrix} a_{(1,1)} & a_{(1,2)} \\ a_{(2,1)} & a_{(2,2)} \end{bmatrix} \vec{x} \tag{1.49}$$

Given this equation we can find the equilibrium point of the system by simply solving this linear set of equations. There are two possible solutions for such a general system, the case where the matrix A is nonsingular and the case where the matrix A is singular. Each of these two cases will be discussed in detail below.

Example 7: Nonsingular Case The simplest case is the nonsingular case where the set of differential equations is non-redundant. In other words the system may be row reduced to the identity matrix, or the determinant of the state matrix is non-zero. Since this matrix can always be reduced to the identity matrix, the trivial solution to the system (i.e., $\vec{x} = 0$) will satisfy the equilibrium equation. In other words, if the system of differential equations is linear and the matrix A is nonsingular, then the only equilibrium point for the system is located at $\vec{x} = 0$. For example, given the system:

$$0 = \mathrm{A}\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x}, \tag{1.50}$$

to show that this system in fact has only one equilibrium point at $\vec{x} = 0$, we simply need to show that the determinant of A is non-zero. Thus,

$$\det(A) = (a_{(1,1)})(a_{(2,2)}) - (a_{(1,2)})(a_{(2,1)}) = (1)(1) - (2)(2) = -3 \neq 0, \tag{1.51}$$

therefore the matrix A has only one equilibrium point located at $\vec{x} = 0$.

Example 8: Forced Nonsingular Case Expanding on the previous example, we can also add a constant forcing function and see how the system equilibrium point changes. For example, let's expand the above example to to be:

$$\dot{\vec{x}} = \mathrm{A}\vec{x} + \mathrm{B}u = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} u. \tag{1.52}$$

Again, to solve for the equilibrium points we set $\dot{\vec{x}} = 0$, thus:

$$0 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} u. \tag{1.53}$$

Now, since we want to analyze this system when it is forced by a constant function, let $u(t)$ be the step function. This allows us to rewrite the above equation as:

$$-\mathrm{B} = \mathrm{A}\vec{x}, \tag{1.54}$$

which can be solved by simply using standard linear algebra techniques. Rewriting the above expression into a augmented matrix and row reducing the resulting expression we find that:

$$\left[\begin{array}{cc|c} 1 & 2 & -3 \\ 2 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right]. \tag{1.55}$$

Thus, the equilibrium point of this system will be $\vec{x} = [1 \;\; -2]^T$ when the forcing function $u(t) \equiv 1$ for all positive time regardless of initial conditions.

Example 9: Singular Case The case in which A is singular shows a very different result. The best way to understand the relationship which develops in a singular state matrix is to go through an example.

Given the linear set of simultaneous differential equations

$$\dot{\vec{x}} = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \vec{x}, \tag{1.56}$$

we want to find the equilibrium points of this matrix. The first step is to rewrite the differential equations in a linear form and set the derivatives to zero as is shown below:

$$0 = x_1 - 2x_2 \tag{1.57}$$

$$0 = 3x_1 - 6x_2. \tag{1.58}$$

When these equations are rearranged and simplified, they both produced the same result:

$$x_1 = 2x_2. \tag{1.59}$$

Therefore there are an infinite number of equilibria which lie along the line described by the equation 1.59. Thus, for any singular state matrix the system will have an infinite number of equilibrium points. The line which describes all possible equilibria is often called the substable manifold.

Although it is important to understand the result of a singular state matrix, it should be noted that these types of systems are rarely seen in naturally occurring phenomena.

Equilibria of Non-Linear Systems When finding equilibria for nonlinear systems, there is no simple, well defined way to find a given systems equilibrium points. Nonlinear systems exhibit much more complex dynamics, as they are not limited to the use of linear functions. Some examples of common nonlinearities in biological systems are transport delays, thresholds, and saturation. In addition, most nonlinear systems will have multiple equilibrium points with vastly varying dynamic profiles.

When solving for equilibria in nonlinear systems often all that is needed is knowledge of algebraic manipulation techniques. There are, however, many more complex methods for finding system equilibria which are outside of the scope of these notes. To illustrate many of the useful techniques for solving for nonlinear equilibria we will find the equilibrium points of two nonlinear systems in examples 10 and 11 below.

Example 10 Given the nonlinear differential equations:

$$\begin{aligned} \dot{x} &= (x-1)(y-1) && (1.60) \\ \dot{y} &= x - y + 3, && (1.61) \end{aligned}$$

where x and y are the two states of the system, solve for the equilibrium states of the system.

Solution. The first thing to do when solving for the equilibrium points is to set the derivative states to zero. Thus the above equations become:

$$\begin{aligned} 0 &= (x-1)(y-1) && (1.62) \\ 0 &= x - y + 3. && (1.63) \end{aligned}$$

Looking at equation , it is obvious that there are two possible values that will make the equation equal to zero: when $x = 1$ and when $y = 1$. Therefore, to find the equilibrium points all we have to do is plug each of these values into equation 1.63 and solve for the remaining variable.

Now, given $x = 1$, we can solve for y using simple algebra:

$$\begin{aligned} 0 &= x - y + 3 && (1.64) \\ 0 &= 1 - y + 3 && (1.65) \\ 4 &= y, && (1.66) \end{aligned}$$

thus one of the equilibrium points for this system is located at $x = 1$ and $y = 4$.

Again, given $y = 1$, we can solve for x using simple algebra:

$$\begin{aligned} 0 &= x - y + 3 && (1.67) \\ 0 &= x - 1 + 3 && (1.68) \\ -2 &= x && (1.69) \end{aligned}$$

thus the other equilibrium point for this system is located at $x = -2$ and $y = 1$.

Example 11 Model a damped simple pendulum as a system of ODEs and find all equilibrium points of the system.

Solution. The first step when analyzing mechanical systems is to draw a free body diagram, as is shown in figure 1.3. The free body diagram explicitly describes the system restoring force, which is in the form of gravitation. Since we are dealing with a rotational system, the fundamental equation we are going to be dealing with is the expression which describes the system torque balance. This equation is often written as:

$$\tau = \mathrm{I}\dot{\omega} + \beta\omega, \qquad (1.70)$$

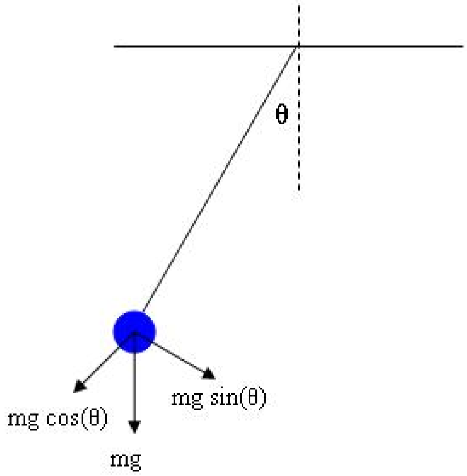


Figure 1.3: Free body diagram showing the forces acting on the pendulum bob as described in example 11.

where τ is the torque on the pendulum, I is the inertial moment of the pendulum, ω is the angular velocity of the pendulum and β is the damping coefficient. The left side of this equation describes the restoring force on the pendulum and the right side of the equation describes the force due to the pendulum's stored kinetic energy as well as the nonconservative damping force found in all realistic systems.

Now that we have established a general equation which describes the system dynamics, we need to find equations which describe the restoring torque on the pendulum as well as the inertial moment of the pendulum. Remembering the general definition of torque:

$$\tau = \tilde{\mathrm{F}} \times \vec{r}, \tag{1.71}$$

where $\tilde{\mathrm{F}}$ is the force vector on the object and $\vec{r}$ is the radius vector from the center of motion to the point at which the force is applied. Using the free body diagram as a guide and noting that the magnitude of the torque can be found by multiplying the magnitude of $\vec{r}$ with the component of force perpendicular to the moment arm, we can represent the torque on the pendulum bob as:

$$\tau = -mgL\sin\theta. \tag{1.72}$$

The inertial moment of the system can be found in any elementary physics text to be:

$$\mathrm{I} = mL^2. \tag{1.73}$$

Plugging the two previous equations into equation 1.70, we find:

$$-mgL\sin\theta = mL^2\dot{\omega} + \beta\omega. \tag{1.74}$$

Rearranging to solve for $\dot{\omega}$, we find that

$$\dot{\omega} = -\frac{g}{L}\sin\theta - \frac{\beta}{mL^2}\omega. \tag{1.75}$$

Finally, we must write the system of differential equations in its final form:

$$\begin{aligned} \dot{\theta} &= \omega && (1.76)\\ \dot{\omega} &= -\frac{g}{L}\sin\theta - \frac{\beta}{mL^2}\omega. && (1.77) \end{aligned}$$

Now to find the equilibrium points, let's simplify the equation by setting all of the constants to one as they become irrelevant. Our differential equations then become:

$$\begin{aligned} \dot{\theta} &= \omega && (1.78)\\ \dot{\omega} &= -\sin\theta - \omega. && (1.79) \end{aligned}$$

When we set the first differential equation to zero, we find that at equilibrium $\omega \equiv 0$. This is intuitive because for the pendulum to be stationary at one point the velocity of the pendulum must be zero. Therefore, to find the equilibria of the above system we must next set the second equation equal to zero while setting ω to zero,

$$\begin{aligned} \dot{\omega} &= -\sin\theta - \omega && (1.80)\\ 0 &= -\sin\theta. && (1.81) \end{aligned}$$

In other words, the system is at equilibrium only when $\omega \equiv 0$ and $\theta \equiv \arcsin 0$, which is true for all values of $\theta = n\pi | n \in Z$. Therefore there are an infinite number of equilibrium points when $\omega = 0$ and θ is an integer multiple of π. This is also intuitive as the pendulum will be either straight down or straight up when θ is an integer multiple of π or 180 degrees.

Linearization and The Jacobian Matrix

One of the more powerful techniques in nonlinear dynamics is the ability to linearize a nonlinear system. In dynamics analysis, linearization is a technique used specifically near equilibrium points to approximate a nonlinear system of ODEs as a linear system. Linearized systems can be used to develop controllers for dynamical systems utilizing well developed linear controller techniques, and to better understand the local behavior of dynamical systems near equilibrium points.

One of the most common linearization techniques utilizes the Taylor series of the dynamical ODEs. Recall that the general form of the Taylor series is:

$$f(x=a) = f(a) + \frac{f_x(a)}{1!}(x-a) + \frac{f_{xx}(a)}{2!}(x-a)^2 + \ldots + \frac{f_{x^n}}{n!}(x-a)^n, \qquad (1.82)$$

where $f(x)$ is an arbitrary function, a is the point where the series is being evaluated, and f_x is the first derivative of the function$f(x)$ with respect to x, f_{xx} is the second derivative, etc. If we ignore the higher order terms of the Taylor series we are left with a linear approximation of the function, $f(x)$:

$$\begin{aligned} f(x=a) &= f(a) &&+ \frac{f_x(a)}{1!}(x-a) \qquad (1.83) \\ y &= b &&+ m(x-a). \qquad (1.84) \end{aligned}$$

We can expand this linear approximation technique further into an analog which we can apply to systems of equations as:

$$\mathrm{J}(\vec{x}) = \mathrm{A}^*(\vec{x}) = \begin{bmatrix} \frac{d\dot{x_1}}{x_1} & \cdots & \frac{d\dot{x_1}}{x_n} \\ \vdots & \ddots & \vdots \\ \frac{d\dot{x_n}}{x_1} & \cdots & \frac{d\dot{x_n}}{x_n} \end{bmatrix}, \qquad (1.85)$$

where J is called the Jacobian matrix, and A* is the state matrix resulting from the linearization of a set of nonlinear ODEs. Now, there are a few *very* important things to note when dealing with linearized systems and the Jacobian:

1. The Jacobian matrix approximation requires that the Jacobian matrix be evaluated at a point in the state space. This is why the J is a function of the state vector, $\vec{x}$.

2. Referring back to equation 1.84, notice that the approximation is a function of $(x-a)$. The equilibrium point in the nonlinear system becomes the origin of the linearized system. For example, if a 2nd-order system is linearized about an equilibrium point at $\vec{x_e} = [3\ 2]^T$, then the point $\vec{x} = [5\ 1]^T$ in the nonlinear system will correspond to the point $\vec{x} = [2\ \ -1]^T$ in the linearized system.

3. The linearized system will reflect the system behavior if and only if the system is linearized around an equilibrium point.

4. The linearized system will accurately represent your nonlinear system only to a degree; if you move too far out of the dynamic range of the linearization it could become very inaccurate.

Next we will go through two examples which show the procedure for linearization of nonlinear differential equations.

Example 12 Linearize the damped pendulum system described in example 11 about the equilibrium point $\vec{x} = [0 \;\; 0]^T$

Solution. Recalling the nonlinear differential equations we established for the pendulum system:

$$\begin{aligned} \dot{\theta} &= \omega && (1.86) \\ \dot{\omega} &= -\sin\theta - \omega, && (1.87) \end{aligned}$$

we must first take all of the derivatives to build the Jacobian matrix (equation 1.85). These are:

$$\begin{aligned} \frac{d\dot{\theta}}{d\theta} &= 0 && (1.88) \\ \frac{d\dot{\theta}}{d\omega} &= 1 && (1.89) \\ \frac{d\dot{\omega}}{d\theta} &= -\cos\theta && (1.90) \\ \frac{d\dot{\omega}}{d\omega} &= -1. && (1.91) \end{aligned}$$

The Jacobian can therefore be written as

$$\mathrm{J}(\vec{x}) = \begin{bmatrix} 0 & 1 \\ -\cos\theta & -1 \end{bmatrix} \text{s.t.} \vec{x} = \begin{bmatrix} \theta \\ \omega \end{bmatrix}. \quad (1.92)$$

The last step is to plug in the linearization point into the equation (i.e., plug in $\vec{x} = [0 \;\; 0]^T$):

$$\mathrm{A}^* = \begin{bmatrix} 0 & 1 \\ -\cos(0) & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}. \quad (1.93)$$

A* describes the system near the equilibrium point when the pendulum is pointing straight down. It is interesting to note that the $\sin\theta$ term in the original set of differential equations was transformed to be simply θ, which reflects the small angle approximation that is often used by physics textbooks for this same problem.

Example 13 Linearize the damped pendulum system described in example 11 about the equilibrium point $\vec{x} = [\pi \;\; 0]^T$.

Solution. Since in the previous example we already found the Jacobian, we simply need to plug the new linearization point into the Jacobian:

$$\mathrm{A}^* = \begin{bmatrix} 0 & 1 \\ -\cos(\pi) & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}. \tag{1.94}$$

Although this may seem like a trivial problem after already doing example 12, it is important to emphasize that the coordinate system has changed in the linear system. Recall that all linear systems have only one equilibrium point located at $\vec{x} = 0$, so our linearized system has moved the point $\vec{x} = [\pi \;\; 0]^T$ to the origin. To alleviate confusion, we often write our linearize systems in terms of error coordinates. We would therefore write our new system as:

$$\begin{bmatrix} \dot{\theta_e} \\ \dot{\omega_e} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \theta_e \\ \omega_e \end{bmatrix}, \tag{1.95}$$

where $\theta_e = \theta - \pi$ and $\omega_e = \omega - 0$.

1.2 Transfer Functions and Block Diagrams

Up to this point we have mainly been representing dynamical systems in the form of time domain systems of ordinary differential equations. There is, however, another way to represent LTI systems using the Laplace transform. Transfer functions which describe the way in which an incoming signal is modified when going through a system can be used to describe the response of the system to an arbitrary input function, $u(t)$. In this section we will first learn how to convert a system represented in state space form into a transfer function and vice versa, as well as some techniques for using block diagrams to create composite transfer functions of multiple systems.

1.2.1 Calculating the Laplace Transfer Function from State Space

To find a general equation to convert a state space system into a transfer function, we must first start with the system representation in state space form:

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u \tag{1.96}$$

$$y(t) = \mathrm{C}x(t) + \mathrm{D}u. \tag{1.97}$$

The other property we must recall is the differentiation property of the Laplace transform:

$$\pounds\left[\dot{f}(t)\right] = sF(s) - f(0). \tag{1.98}$$

Using this property and taking the Laplace transform of the general state space form we find that:

$$\begin{aligned} sX(s) &= \mathrm{A}X(s) + \mathrm{B}U(s) && (1.99)\\ Y(s) &= \mathrm{C}X(s) + \mathrm{D}U(s). && (1.100) \end{aligned}$$

Since the general form of the transfer function is $Y(s)/U(s)$, we can rewrite the above equations to satisfy the transfer function form:

$$\begin{aligned} sX(s) &= \mathrm{A}X(s) + \mathrm{B}U(s) && (1.101)\\ sX(s) - \mathrm{A}X(s) &= \mathrm{B}U(s) && (1.102)\\ (s\,\mathrm{I} - \mathrm{A})X(s) &= \mathrm{B}U(s) && (1.103)\\ X(s) &= (s\,\mathrm{I} - \mathrm{A})^{-1}\,\mathrm{B}U(s), && (1.104) \end{aligned}$$

substituting into the output equation:

$$\begin{aligned} Y(s) &= \mathrm{C}X(s) + \mathrm{D}U(s) && (1.105)\\ Y(s) &= \mathrm{C}(s\,\mathrm{I} - \mathrm{A})^{-1}\,\mathrm{B}U(s) + \mathrm{D}U(s) && (1.106)\\ \frac{Y(s)}{U(s)} &= \mathrm{C}(s\,\mathrm{I} - \mathrm{A})^{-1}\,\mathrm{B} + \mathrm{D}. && (1.107) \end{aligned}$$

Using equation 1.107, the transfer function of any system represented in state space can be converted into a transfer function. It is also important to note that the expression $s\,\mathrm{I} - \mathrm{A}$ is very similar to the equation used to find the eigenvalues of a matrix, $det(\lambda\,\mathrm{I} - A)$. This similarity explains why the characteristic equation of the system as described by the state matrix eigen-structure also, as we will see, describes the pole structure in the Laplace domain. In other words, this similarity shows how both system representations contain the same information.

Example 14 Find the transfer function for the following system:

$$\dot{\vec{x}} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad (1.108)$$

$$y = [1\ \ 1]\,\vec{x}. \qquad (1.109)$$

Solution. Plugging the above matrices into equation 1.107, we must solve:

$$\frac{Y}{U} = [1\ \ 1] \begin{bmatrix} s-1 & -2 \\ -2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 0 \end{bmatrix}. \qquad (1.110)$$

The first step is to invert the matrix $s\,\mathrm{I} -\ \mathrm{A}$. Since the matrix is 2-by-2 we can use the following equation to find the inverse of the matrix:

$$M^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{det(M)} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}. \tag{1.111}$$

Therefore, the inverse of the matrix $s\,\mathrm{I} -\ \mathrm{A}$ is:

$$\begin{bmatrix} s-1 & -2 \\ -2 & s-1 \end{bmatrix}^{-1} = \frac{1}{s^2-2s-3} \begin{bmatrix} s-1 & 2 \\ 2 & s-1 \end{bmatrix}. \tag{1.112}$$

Replacing this in equation 1.110:

$$\frac{Y}{U} = \frac{1}{s^2-2s-3} [1\ \ 1] \begin{bmatrix} s-1 & 2 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}. \tag{1.113}$$

Multiplying the matrices we find the transfer function to be:

$$\frac{Y}{U} = \frac{3s+3}{s^2-2s-3}. \tag{1.114}$$

One thing to note again is that the characteristic equation of the system can be found in the denominator of the transfer function. Thus, the characteristic equation will define the location of the system poles in the Laplace domain, which will determine much of the behavior of the system response.

1.2.2 Block Diagrams

One of the most common way complex systems with various components are modeled as a system is in the form of block diagrams. An example of a block diagram is shown in figure 1.4. Block diagrams are designed to show the connectivity of multiple subsystems (such as physical system models, controllers, filters, etc.) which

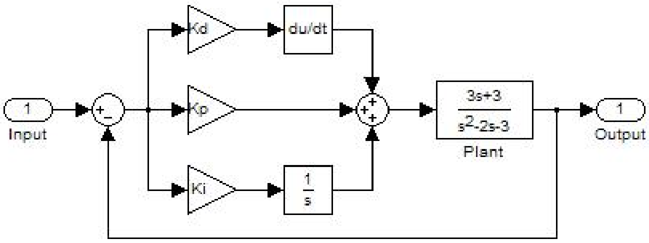


Figure 1.4: A PID controller of a system (the plant) including negative feedback.

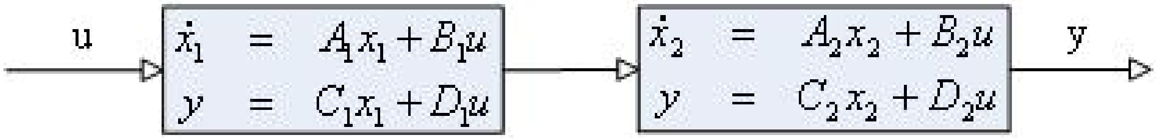


Figure 1.5: A generic series connected system.

compose a larger system. It is often very useful to combine multiple subsystems into a single system allowing us to better understand how the entire system works as a whole. Being able to understand how various subsystems interact with one-another and create an overall dynamic is crucial to understand the ways in which we can create subsystems which will modify the overall system dynamics in a way that is useful from an engineering perspective. That is, it allows us to understand the ways in which we can take a physical system and modify it's behavior to make it work in a useful way. In this section, we will discuss multiple ways to combine systems in both the time and Laplace domains.

Series Connections

Series connectivity is seen very often. Series connected subsystems are wired such that output of the first system is fed directly into the input of the next, forming a chain. The mathematics behind combining series subsystems is discussed below.

State Space A generic set of series connected state space systems is shown in figure 1.5. To combine these two systems, we must rewrite the input of the second system, H, to be dependent on the previous system, G. Therefore, we can state that:

$$u_2 = \mathrm{C}_1\vec{x}_1. \tag{1.115}$$

Therefore, rewriting the differential equations and ignoring D (as it is usually 0) describing the entire system we find that:

$$\begin{aligned} \dot{\vec{x}}_1 &= \mathrm{A}_1x_1 + \mathrm{B}_1u & (1.116)\\ \dot{\vec{x}}_2 &= \mathrm{A}_2x_2 + \mathrm{B}_2\,\mathrm{C}_1x_1 & (1.117)\\ y &= \mathrm{C}_2x_2, & (1.118) \end{aligned}$$

thereby eliminating the intermediate equation which describes the output of the first system. The second trick to combining these systems is to combine the two vectors $\vec{x}_1$ and $\vec{x}_2$ into a single vector. By doing this we can now write the system in a single form:

$$\begin{bmatrix} \dot{\vec{x}}_1 \\ \dot{\vec{x}}_2 \end{bmatrix} = \left[\begin{array}{c|c} \mathrm{A}_1 & 0 \\ \hline \mathrm{B}_2\,\mathrm{C}_1 & \mathrm{A}_2 \end{array}\right] \begin{bmatrix} \vec{x}_1 \\ \vec{x}_2 \end{bmatrix} + \begin{bmatrix} \mathrm{B}_1 \\ 0 \end{bmatrix} u \tag{1.119}$$

$$y = [0 \;\; \mathrm{C}_2]\,[\vec{x}_1 \;\; \vec{x}_2]^T. \tag{1.120}$$

Example 15 Combine the two following systems in state space form as if they have series connectivity, and then find the system transfer function:

$$\mathrm{A}_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \mathrm{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathrm{C}_1 = [1 \;\; 0]\,, \quad \text{and } \mathrm{D}_1 = 0; \tag{1.121}$$

$$\mathrm{A}_2 = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \quad \mathrm{B}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathrm{C}_2 = [2 \;\; 1]\,, \quad \text{and } \mathrm{D}_2 = 0; \tag{1.122}$$

Solution. Given the two systems and equation 1.119 above, we can simply do the matrix multiplication steps and rewrite the above systems as:

$$\mathrm{A}_3 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 4 \\ 1 & 0 & 1 & 2 \end{bmatrix}, \quad \mathrm{B}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathrm{C}_3 = [0 \;\; 0 \;\; 2 \;\; 1]\,. \tag{1.123}$$

Using equation 1.107, we find the transfer function of the above system to be:

$$T(s) = \frac{6s+4}{s^4 - 7s^3 + 9s^2 + 11s - 6}. \tag{1.124}$$

Transfer Functions Working with block diagrams using transfer functions are, in most cases, much easier than working in state space. Series connected systems are no exception. When combining two series connected systems, you simply need to multiply the transfer functions of each of the series connected systems together. To prove this we will assume this is true and prove that when you multiply these two systems together the resulting transfer function will be Ys/Us.

Given the series system shown in figure 1.6, we can write the transfer function of each system as:

$$G(s) \;\; = \;\; \frac{A(s)}{U(s)} \tag{1.125}$$

$$H(s) \;\; = \;\; \frac{Y(s)}{A(s)}, \tag{1.126}$$

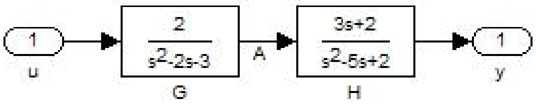


Figure 1.6: An example of a feedback connected system.

where $A(s)$ is the intermediate function which describes the output of the system G. If we multiply the two signals together:

$$G(s)H(s) = \left(\frac{A(s)}{U(s)}\right)\left(\frac{Y(s)}{A(s)}\right) = \frac{Y(s)}{U(s)}, \tag{1.127}$$

which is the definition of the system transfer function.

Example 16 Confirm that the above rule for combining transfer functions applies to the systems in example 1.3.2 (figure 1.6).

Solution. First, we must use equation 1.107 to each of the above systems into transfer functions. If we let the first system to be signified by g and the second system by h we find:

$$G(s) = \mathrm{C}_1(s\,\mathrm{I} - \mathrm{A}_1)^{-1}\,\mathrm{B}_1 = \frac{2}{s^2 - 2s - 3} \tag{1.128}$$

$$H(s) = \mathrm{C}_2(s\,\mathrm{I} - \mathrm{A}_2)^{-1}\,\mathrm{B}_2 = \frac{3s + 2}{s^2 - 5s + 2}. \tag{1.129}$$

Applying the rule for combining transfer functions of series systems:

$$G(s)H(s) = \left(\frac{2}{s^2 - 2s - 3}\right)\left(\frac{3s + 2}{s^2 - 5s + 2}\right) = \frac{6s + 4}{s^4 - 7s^3 + 11s - 6}, \tag{1.130}$$

which is the same as the solution to example 1.3.1, showing that the two methods are equivalent.

Parallel Connections

The other type of common simple connectivity is the parallel connection. Parallel subsystems share the same input and sum their outputs. Equations for combining parallel subsystems in state space and in the Laplace domain are described below.

State Space A generic set of parallel connected subsystems is shown in figure **??**. As was done in the serial case, we must rewrite the system equations in a form which relates the connectivity of the two systems.

$$\dot{\vec{x}}_1 = \mathrm{A}_1\vec{x}_1 + \mathrm{B}_1 u \tag{1.131}$$

$$\dot{\vec{x}}_2 = \mathrm{A}_2\vec{x}_2 + \mathrm{B}_2 u \tag{1.132}$$

$$y = \mathrm{C}_1\vec{x}_1 + \mathrm{C}_2\vec{x}_2. \tag{1.133}$$

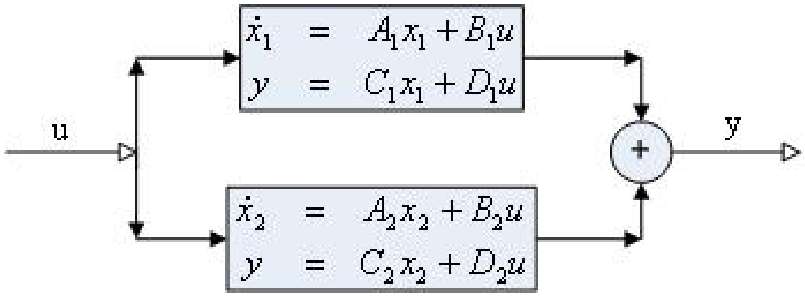


Figure 1.7: A generic parallel connected system.

From these equations it is simple to write the combined system in state space form:

$$\left[\begin{array}{c}\dot{\vec{x}}_1\\ \dot{\vec{x}}_2\end{array}\right] = \left[\begin{array}{c|c}\mathrm{A}_1 & 0\\ \hline 0 & \mathrm{A}_2\end{array}\right]\left[\begin{array}{c}\vec{x}_1\\ \vec{x}_2\end{array}\right] + \left[\begin{array}{c}\mathrm{B}_1\\ \mathrm{B}_2\end{array}\right]u, \tag{1.134}$$

$$y = [\,\mathrm{C}_1 \;\; \mathrm{C}_2]\,[\vec{x}_1 \;\; \vec{x}_2]^T. \tag{1.135}$$

Example 17 Combine the two following systems in state space form as if they have series connectivity, and then find the system transfer function:

$$\mathrm{A}_1 = \left[\begin{array}{cc}1 & 2\\ 2 & 1\end{array}\right], \quad \mathrm{B}_1 = \left[\begin{array}{c}0\\ 1\end{array}\right], \quad \mathrm{C}_1 = [1 \;\; 0]\,, \quad \text{and } \mathrm{D}_1 = 0; \tag{1.136}$$

$$\mathrm{A}_2 = \left[\begin{array}{cc}3 & 4\\ 1 & 2\end{array}\right], \quad \mathrm{B}_2 = \left[\begin{array}{c}1\\ 1\end{array}\right], \quad \mathrm{C}_2 = [2 \;\; 1]\,, \quad \text{and } \mathrm{D}_2 = 0; \tag{1.137}$$

Solution. Given the two systems and equation 1.134 above, we can rewrite the above systems as:

$$\mathrm{A}_3 = \left[\begin{array}{cccc}1 & 2 & 0 & 0\\ 2 & 1 & 0 & 0\\ 0 & 0 & 3 & 4\\ 0 & 0 & 1 & 2\end{array}\right], \quad \mathrm{B}_3 = \left[\begin{array}{c}0\\ 1\\ 1\\ 1\end{array}\right], \quad \mathrm{C}_3 = [1 \;\; 0 \;\; 2 \;\; 1]\,. \tag{1.138}$$

Using equation 1.107, we find the transfer function of the above system to be:

$$T(s) = \frac{3s^3 - 2s^2 - 23s - 2}{s^4 - 7s^3 + 9s^2 + 11s - 6}. \tag{1.139}$$

Transfer Functions The transfer function representation of the feedback system is shown in figure 1.8. Again, we want to solve for the transfer function with the form $Y(s)/U(s)$. We may start by writing the I/O expressions for the two subsystems:

$$\begin{aligned} A_1(s) &= G(s)U(s) && (1.140) \\ A_2(s) &= H(s)U(s), && (1.141) \end{aligned}$$

and the output:

$$Y(s) = A_1(s) + A_2(s). \qquad (1.142)$$

Substituting we find that:

$$Y(s) = G(s)U(s) + H(s)U(s). \qquad (1.143)$$

Rearranging we find

$$\begin{aligned} Y &= (G+H)U && (1.144) \\ \text{or } \frac{Y}{U} &= G + H. && (1.145) \end{aligned}$$

In other words, parallel transfer functions may be combined by simply adding their respective transfer functions.

Example 18 Confirm that the above rule for combining transfer functions applies to the systems in example 1.3.4 (figure 1.8).

Solution. First, we must use equation 1.107 to each of the above systems into transfer functions. If we let the first system to be signified by g and the second system by h we find:

$$\begin{aligned} G(s) &= \mathrm{C}_1(s\,\mathrm{I} - \mathrm{A}_1)^{-1}\,\mathrm{B}_1 = \frac{2}{s^2 - 2s - 3} && (1.146) \\ H(s) &= \mathrm{C}_2(s\,\mathrm{I} - \mathrm{A}_2)^{-1}\,\mathrm{B}_2 = \frac{3s+2}{s^2 - 5s + 2}. && (1.147) \end{aligned}$$

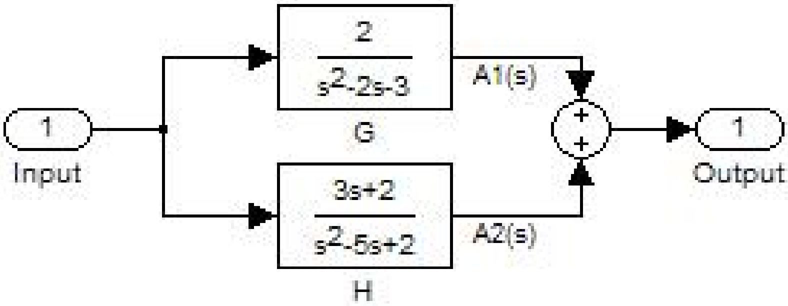


Figure 1.8: An example of a feedback system.

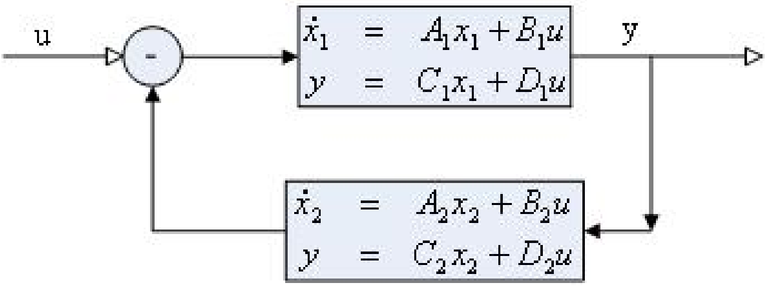


Figure 1.9: A generic negative feedback system.

Applying the rule for combining parallel systems we find that:

$$T(s) = \frac{2}{s^2 - 2s - 3} + \frac{3s + 2}{s^2 - 5s + 2}. \tag{1.148}$$

Finding the common denominator,

$$T(s) = \frac{(2)(s^2 - 5s + 2) + (3s + 2)(s^2 - 2s - 3)}{(s^2 - 2s - 3)(s^2 - 5s + 2)} \tag{1.149}$$

and simplifying

$$\begin{aligned} T(s) &= \frac{2s^2 - 10s + 4 + 3s^3 - 4s^2 - 13s - 6}{s^4 - 7s^3 + 9s^2 + 11s - 6} && (1.150) \\ T(s) &= \frac{3s^3 - 2s^2 - 23s - 2}{s^4 - 7s^3 + 9s^2 + 11s - 6}, && (1.151) \end{aligned}$$

which is the same as the solution to example 1.3.4.

Negative Feedback Connections

Negative feedback systems are the cornerstone of modern linear control theory. Feedback systems are found in many complicated dynamical systems including physiological and biochemical systems, operational amplifers and mechanical governors, just to name a few. The mathematics of combining negative feedback systems is described below.

State Space A generic set of feedback connected state space systems is shown in figure 1.9. Similar to the techniques described above, we want to rewrite the input and output of each subsystem in terms of known quantities. Therefore, from the block diagram we can see that:

$$\begin{aligned} u_1 &= u - \mathrm{C}_2\vec{x}_2 && (1.152) \\ \text{and } u_2 &= \mathrm{C}_1\vec{x}_1. && (1.153) \end{aligned}$$

Rewriting the state space equations, we find that:

$$\begin{aligned}
\dot{\vec{x}}_1 &= \mathrm{A}_1\vec{x}_1 + \mathrm{B}_1(u - \mathrm{C}_2\vec{x}_2) = \mathrm{A}_1\vec{x}_1 - \mathrm{B}_1\,\mathrm{C}_2\vec{x}_2 + \mathrm{B}_1 u && (1.154)\\
\dot{\vec{x}}_2 &= \mathrm{A}_2\vec{x}_2 + \mathrm{B}_2(\,\mathrm{C}_1\vec{x}_1) && (1.155)\\
y &= \mathrm{C}_1\vec{x}_1. && (1.156)
\end{aligned}$$

Finally, this can be written in state space form as:

$$\begin{bmatrix} \dot{\vec{x}}_1 \\ \dot{\vec{x}}_2 \end{bmatrix} = \left[\begin{array}{c|c} \mathrm{A}_1 & -\,\mathrm{B}_1\,\mathrm{C}_2 \\ \hline \mathrm{B}_2\,\mathrm{C}_1 & \mathrm{A}_2 \end{array}\right] \begin{bmatrix} \vec{x}_1 \\ \vec{x}_2 \end{bmatrix} + \begin{bmatrix} \mathrm{B}_1 \\ 0 \end{bmatrix} u \qquad (1.157)$$

$$y = [\,\mathrm{C}_1 \;\; 0]\,[\vec{x}_1 \;\; \vec{x}_2]^T. \qquad (1.158)$$

Example 19 Combine the two following systems in state space form as if they have series connectivity, and then find the system transfer function:

$$\mathrm{A}_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \mathrm{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathrm{C}_1 = [1 \;\; 0]\,, \quad \text{and } \mathrm{D}_1 = 0; \qquad (1.159)$$

$$\mathrm{A}_2 = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \quad \mathrm{B}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathrm{C}_2 = [2 \;\; 1]\,, \quad \text{and } \mathrm{D}_2 = 0; \qquad (1.160)$$

Solution. Given the two systems and equation 1.119 above, we can simply do the matrix multiplication steps and rewrite the above systems as:

$$\mathrm{A}_3 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & -2 & -1 \\ 1 & 0 & 3 & 4 \\ 1 & 0 & 1 & 2 \end{bmatrix}, \quad \mathrm{B}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathrm{C}_3 = [1 \;\; 0 \;\; 0 \;\; 0]\,. \qquad (1.161)$$

Using equation 1.107, we find the transfer function of the above system to be:

$$T(s) = \frac{2s^2 - 10s + 4}{s^4 - 7s^3 + 9s^2 + 17s - 2}. \qquad (1.162)$$

Transfer Functions The transfer function representation of the feedback system is shown in figure 1.10. To find the combined transfer function for this system, some mathematical manipulation will need to be done on each of intermediate signals E(s) and A(s). The fundamental equations for this system are:

$$\begin{aligned}
E &= U - A && (1.163)\\
Y &= EG && (1.164)\\
A &= YH. && (1.165)
\end{aligned}$$

Next, we can plug E into the equation which expresses Y:

$$Y = EG = (U - A)G. \tag{1.166}$$

Plugging the expression for A into this equation we find that:

$$\begin{aligned} Y &= (U - YH)G \quad (1.167) \\ \text{or } \frac{Y}{U} &= \frac{G}{1 + GH}, \quad (1.168) \end{aligned}$$

which is the transfer function for the combined system.

Example 20 Confirm that the above rule for combining transfer functions applies to the systems in example 1.3.6 (figure 1.10).

Solution. First, we must use equation 1.107 to each of the above systems into transfer functions. If we let the first system to be signified by g and the second system by h we find:

$$\begin{aligned} G(s) &= \mathrm{C}_1(s\,\mathrm{I} - \mathrm{A}_1)^{-1}\,\mathrm{B}_1 = \frac{2}{s^2 - 2s - 3} \quad (1.169) \\ H(s) &= \mathrm{C}_2(s\,\mathrm{I} - \mathrm{A}_2)^{-1}\,\mathrm{B}_2 = \frac{3s + 2}{s^2 - 5s + 2}. \quad (1.170) \end{aligned}$$

Applying equation 1.168 to combining parallel systems we find that:

$$T(s) = \frac{G}{1 + GH} = \frac{\frac{2}{s^2-2s-3}}{1 + \left(\frac{2}{s^2-2s-3}\right)\left(\frac{3s+2}{s^2-5s+2}\right)}. \tag{1.171}$$

Next we multiply by the denominator of the GH term on the top and bottom of the

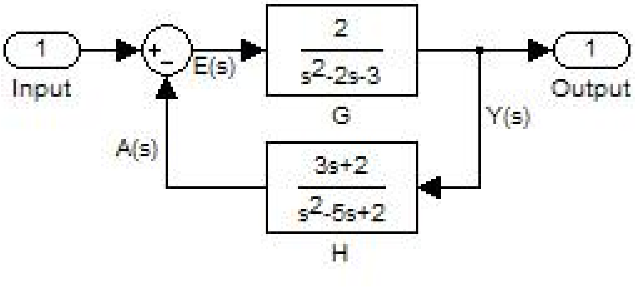


Figure 1.10: An example of a feedback system.

main fraction:

$$T(s) = \frac{\frac{2}{s^2-2s-3}}{1+\left(\frac{2}{s^2-2s-3}\right)\left(\frac{3s+2}{s^2-5s+2}\right)}\frac{(s^2-2s-3)(s^2-5s+2)}{(s^2-2s-3)(s^2-5s+2)} \quad (1.172)$$

$$T(s) = \frac{(2)(s^2-5s+2)}{(s^2-2s-3)(s^2-5s+2)+(2)(3s+2)}, \quad (1.173)$$

expanding we find:

$$T(s) = \frac{2s^2-10s+4}{s^4-7s^3+9s^2+17s-2}, \quad (1.174)$$

which is the same result as example 1.3.6.

1.3 Supplemental: Finding State Space Equations from a Transfer Function

As we saw before, it is relatively simple to convert a system represented in state space form into a transfer function. The inverse problem, however, is not so simple. Since there are an infinite number of state space equations which can be used to represent the same system there is no set way to reproduce the system using the variables we may discern from a physical system. Although it is outside the scope of this text to describe the theory behind the transform from a transfer function back into state space form, in this section a few common conversions will be outlined. The theory behind these transforms can be found in any linear control theory text.

1.3.1 Controller Canonical Form

To convert the generic transfer function:

$$T(s) = \frac{b_2s^2+b_1s+b_0}{s^3+a_2s^2+a_1s+a_0} \quad (1.175)$$

to the controller canonical form, we simply write the state space equations as:

$$\mathrm{A} = \begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathrm{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathrm{C} = [b_2 \ \ b_1 \ \ b_0]\,. \quad (1.176)$$

Note that this pattern will scale with the order of the system.

1.3.2 Observer Canonical Form

To convert the generic transfer function:

$$T(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} \tag{1.177}$$

to the observer canonical form, we simply write the state space equations as:

$$\mathrm{A} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}, \quad \mathrm{B} = \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}, \quad \mathrm{C} = [1 \;\; 0 \;\; 0]\,. \tag{1.178}$$

Note that this pattern will scale with the order of the system.

Chapter 2

LTI Dynamics

Now that the basic mathematical constructs used to describe linear dynamics has been developed in the previous chapter, we may begin to analyze the properties of real systems. In this chapter we will discuss many of the ways in which systems will respond to input stimuli with an emphasis on the second order system. The impulse and step response techniques will be described as well as phase plane analysis. Performance metrics of the second order system's step response will be discussed as well due to its significance in control theory. The primary analysis tool used in this section is Matlab using the control systems toolbox. In addition to the above sections, a suplemental section describing the solution method for systems of ODEs using matrix exponentials is included for those students are especially interested in analytical solutions to these systems.

2.1 LTI System Natural Response

In this section, we will begin to learn the fundamental tools in which LTI systems in state-space form are analyzed. Both graphical and quantitative methods for analysis of the natural response of both linear and nonlinear systems will be discussed.

2.1.1 System Natural Response

Recall, that a general function represented in state spate space form has the form:

$$\dot{\vec{x}} = \mathrm{A}\vec{x} + \mathrm{B}u. \tag{2.1}$$

Given an arbitrary system (for the sake of this example, we will use a second order system), the natural response of the above system is:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \mathrm{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathrm{B} \cdot 0. \tag{2.2}$$

Recall that for a differential equation to be true, the general solution to a differential equation takes the form

$$x(t) = e^{rt}, \tag{2.3}$$

since the natural exponential function is the only function which is directly proportional to its derivative. Therefore, we can rewrite equation 2.2 to include equation 2.3, which will give us the natural response solution to the linear set of differential equations:

$$\begin{bmatrix} r \cdot e^{rt} \\ r \cdot e^{rt} \end{bmatrix} = \mathrm{A} \begin{bmatrix} e^{rt} \\ e^{rt} \end{bmatrix}. \tag{2.4}$$

Simplifying, we find:

$$r \begin{bmatrix} e^{rt} \\ e^{rt} \end{bmatrix} = \mathrm{A} \begin{bmatrix} e^{rt} \\ e^{rt} \end{bmatrix}. \tag{2.5}$$

If we now substitute $x(t)$ back into the equation, we find that :

$$r \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathrm{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2.6}$$

or:

$$r\vec{x} = \mathrm{A}\vec{x}, \tag{2.7}$$

which has the exact same form as the eigenvalue equation for an arbitrary matrix, M:

$$\lambda\vec{x} = \mathrm{M}\vec{x}, \tag{2.8}$$

where λ an eigenvalue of the matrix, M. Therefore, we can solve for the decay constants for the natural response using the following equation:

$$0 = (r\,\mathrm{I} - \mathrm{A})\vec{x}. \tag{2.9}$$

Applying Cramer's rule which states that the above equation only has non-trivial solutions when the determinant of the matrix term is zero, the above equation is simplified to:

$$0 = det(r\,\mathrm{I} - \mathrm{A}). \tag{2.10}$$

This equation will produce a polynomial of order equal to the number of system states in r. There will, therefore, be multiple unique solutions to the linear set of

ODEs. The overall system response, therefore, is a linear combination of all the possible solutions to the set of linear ODEs:

$$y(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + \cdots + a_n e^{\lambda_n t}, \tag{2.11}$$

where n is the number of system states, λ_1 through λ_n are the eigenvalues of the state matrix, and a_1 through a_n are pre-exponential constant which describes the relative weights of each possible solution with respect to the initial conditions of the system. Calculation of the pre-exponential constants and the forced system response is described in section 2.3 below.

One important thing to note is that the determinant function shown in equation 2.10 mirrors the denominator of the transfer function equation derived in equation 1.107. Therefore, the pole locations of the transfer function in the Laplace domain are equivalent to the eigenvalues calculated using the method described above.

Eigenvalues and Stability

Since we proved that the eigenvalues correspond to the exponential constants which constitute the system's natural response (see equation 2.11), we may infer many of the properties of the system's solutions. Since the equation:

$$y(t) = e^{\lambda t} \tag{2.12}$$

is only stable when the real component of the exponent is negative and since we are dealing with a causal system (i.e., $t \geq 0$), we may infer that a system will have a stable natural responses if and only if the real component all eigenvalues of A have negative real components.

Stable, System Impulse Response

Given the general solution to the linear set of differential equations, the type of dynamics observed can also be inferred based on the eigenvalues of the state matrix. Here we will analyze various second order systems (for simplicity) which exhibit the fundamental system response types. Note that the systems described in this section will be stable only, but it is also possible to have similar responses with unstable behavior.

Over Damped Over damped behavior occurs when both eigenvalues of the second order system have only real components, and do not have the same numerical value.

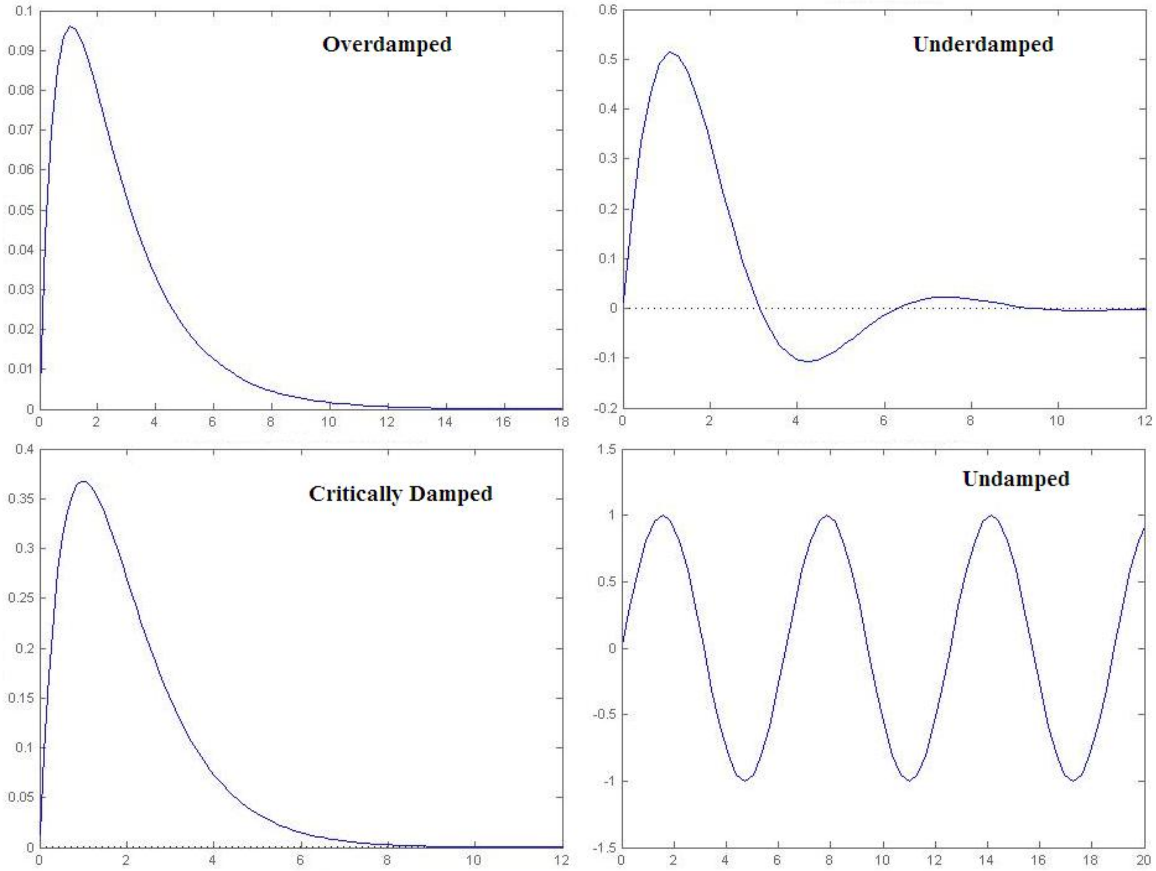


Figure 2.1: Impulse responses for various state-space eigenstructures.

For example, the following system shows over damped behavior:

$$\dot{\vec{x}} = \begin{bmatrix} -1 & 1 \\ .25 & -1 \end{bmatrix} \vec{x}. \tag{2.13}$$

Calculating the eigenvalues,

$$0 = det(\lambda \mathrm{I} - A) = (\lambda + 1)(\lambda + 1) - .25, \;\; \lambda = -.5, \; -1.5. \tag{2.14}$$

The resulting impulse response for this system is shown in figure 2.1.

Underdamped Underdamped behavior occurs when both eigenvalues of the second order system have real and imaginary components, producing an oscillatory response. This is because imaginary components within the exponential translate into a sinusoidal response due to the Euler relation. For example, the following system shows over-damped behavior:

$$\dot{\vec{x}} = \begin{bmatrix} -.5 & -1 \\ 1 & -.5 \end{bmatrix} \vec{x}. \tag{2.15}$$

Calculating the eigenvalues,

$$0 = det(\lambda \mathrm{I} - A) = (\lambda + .5)(\lambda + .5) + 1, \;\; \lambda = -.5 \pm j. \tag{2.16}$$

The resulting impulse response for this system is shown in figure 2.1.

Undamped A special case of the underdamped responses type occurs when the eigenvalues have no real component. As per Euler's relation, there will be a sinusoidal component to the system. The lack of a real component, however, takes away the term described by a decaying exponential. Therefore, the system will oscillate without any loss in overall energy to the system. In this case, the system is often called 'conditionally stable' since the system will reach an infinite value if and only if system input is infinite, but doesn't damp out to a single steady state. For example, the following system shows undamped behavior:

$$\dot{\vec{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}. \tag{2.17}$$

Calculating the eigenvalues,

$$0 = det(\lambda \mathrm{I} - A) = \lambda^2 + 1, \;\; \lambda = \pm j. \tag{2.18}$$

The resulting impulse response for this system is shown in figure 2.1.

Critically Damped Another special case is when the eigenvalues are real and numerically equivalent. This is manifested as a slight modification to equation 2.11, which becomes for a second order critically damped system:

$$y(t) = a_1 e^{\lambda t} + a_2 t e^{\lambda t}, \tag{2.19}$$

where λ is the single eigenvalue of the system. The addition of the t in the second term of the solution allows for the contributions two states to be non-redundantly represented in the solution. For more information regarding this term, see the discussion of matrix exponentials in section 2.3.

An example of a critically damped system is:

$$\dot{\vec{x}} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \vec{x}. \tag{2.20}$$

Calculating the eigenvalues,

$$0 = det(\lambda \mathrm{I} - A) = (\lambda + 1)(\lambda + 1) + 0, \;\; \lambda = -1. \tag{2.21}$$

The resulting impulse response for this system is shown in figure 2.1.

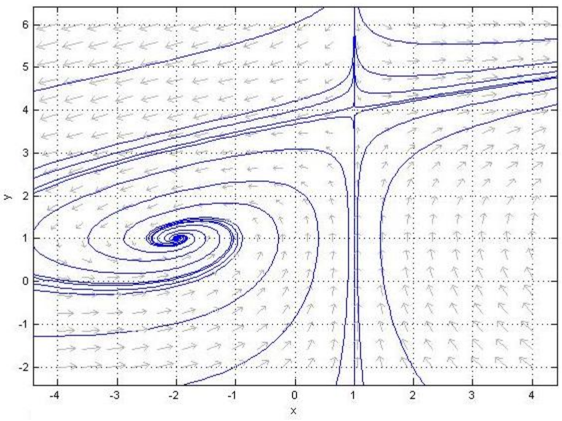


Figure 2.2: The phase portrait of the system described in example 1.1.10.

Eigenvalue Analysis of Nonlinear Systems

Similar system analysis can be executed to better understand nonlinear systems as well using the linearization techniques discussed in the previous chapter. This type of analysis must be done near an equilibrium point, because a system linearized around an arbitrary point will always shown unstable behavior. Thus the local behavior of the system can simply be found by linearizing the system around its equilibrium point(s) and analyzing the eigenstructure of the linearized matrix.

2.1.2 Phase Plane Analysis of 2^{nd} Order Systems

Phase portraits are another graphical method for describing the internal dynamics of a second order system. Instead of simply plotting the response of a system based on its initial conditions or its impulse response, we will begin to explore the ways in which the states of a second order system respond together to a stimulus.

An example of a nonlinear phase portrait is shown in figure 2.22. Notice that there are two components to the phase portrait; solid lines represent what is called a phase trajectory and the grey arrows represent the vector field which describes the pathways which the trajectories follow. Since our system, when represented in state space form, is represented as a derivative vector (i.e., the direction vector of the system state) versus the state vector itself (i.e., the position in the phase plane), a vector field can be created which describes all solutions to the state space model for arbitrary initial conditions.

Since we are analyzing a model in state space form, we are able to monitor the internal states of the solution as they progress through time. For example, the two

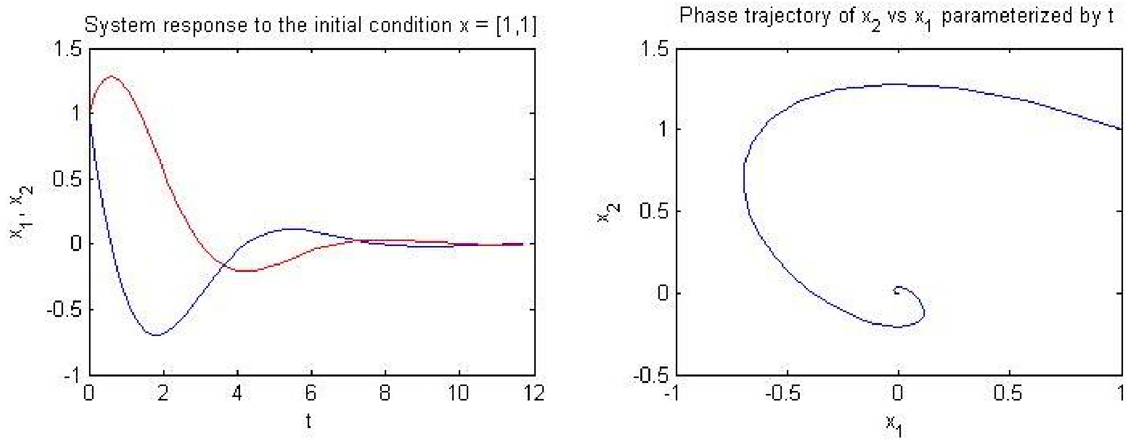


Figure 2.3: Left, the solutions to the initial value problem described in equation 2.22; right, the resulting phase trajectory for the system.

solutions for the system:

$$\dot{\vec{x}} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u; y = [0\ \ 1]\vec{x}, \tag{2.22}$$

with initial conditions $\vec{x} = [1,1]^T$ is shown in figure 2.3 on the left. To create a single phase trajectory, plot one phase variable against the other (i.e., x_2 vs. x_1, as is shown in figure 2.3 on the right. Another way to think of a phase trajectory is if you were to drop a buoy into a tank with water flowing with a velocity profile equal to the systems vector field, the phase trajectory will be the path through which the buoy follows. Often multiple of these trajectories are superimposed on the vector field to help show the behavior of the model as a whole, as is shown in figure 2.22.

Eigenvalues and Linear Phase Portraits

Since phase portraits are a graphical expression of the solution to a system of differential equations, it is logical that the eigenstructure patterns we have previously discussed would be reflected as patterns in a phase portrait as well. The most common patterns are shown in figure 2.4. These patterns constitute almost all of the dynamics shown by linear systems, with one notable exception.

Saddle Points The first dynamic structure we will discuss is the saddle point. Saddle points occur when one of the of the state matrix eigenvalues is negative and the other is positive, creating a pseudostable system where there is an equilibrium point surrounded by a region of unstable behavior. Thus, the saddle point shown

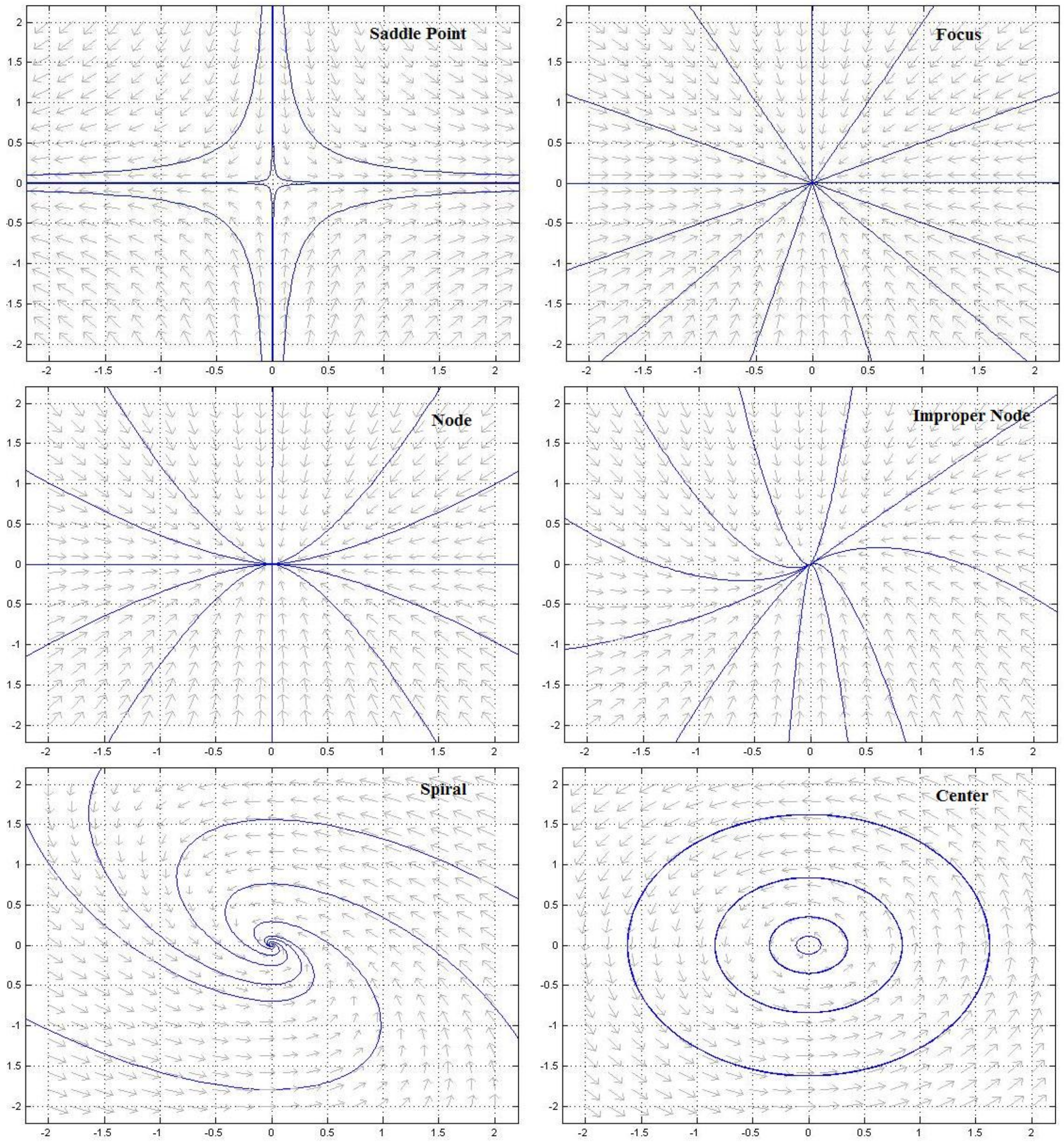


Figure 2.4: Linear phase portraits for various state-space eigenstructures.

in figure 2.4 has a stable point at the origin ($\vec{x} = [0\ \ 0]^T$), and is unstable for any other value of $\vec{x}$, as is shown by the divergent vector field and phase trajectories. A notable example of a saddle point is when a pendulum is placed straight up and down, since it is possible to balance the pendulum straight up but if you nudge it in either direction it will fall away from the equilibrium point.

An interesting thing to note is that saddle points are often rotated about their equilibrium point. The degree of rotation can be found by calculating the angles formed by the eigenvectors of the state matrix.

Centers Centers are another common dynamical archetype which occur when the eigenvalues of the state matrix lie on the imaginary axis (i.e., the eigenvalues have no real component). When the eigenvalues of the system lie on the imaginary axis the dynamical result is that they system will undergo undamped oscillations. Thus the energy in the system will alternate between the two states. In the phase plane this creates a circular structure, as is shown in figure 2.4. A notable example of a center is the oscillation of an undamped pendulum.

Types of Nodes Nodes are a subset of phase portrait patterns which describe dynamical systems which exhibit damped behavior. Depending on the sign of the eigenvalues of the nodal system, these systems can be either stable (phase trajectories tend towards the equilibrium point) or unstable (phase trajectories tend away from the equilibrium point). Stable equilibrium points are often called sinks and unstable equilibrium points are often called sources. The subtle differences between these systems are divided into four sub-categories and are described below.

Foci Foci are exhibited by systems with two equal, real eigenvalues (i.e., a critically damped system), in which the energy contained in the system is damped in the most efficient way possible (for a stable system, that is). Thus, all phase trajectories will take a straight path directly to the system equilibrium point, as is shown in figure 2.4.

Nodes Nodes are very similar to foci; however instead of the system being critically damped the system is over-damped with two eigenvalues on the real axis but with different numerical values. Thus, trajectories for a stable system will tend toward the equilibrium point but will have a noticeable curvature, as is shown in figure 2.4.

Improper Nodes Improper nodes are a subset of aforementioned nodal description which arise when there is a damping factor in one of the differential equations but not the other. This will produce a noticeable skew in the nodal structure, which is shown in figure 2.4. An example of a state matrix which would produce an improper node would be:

$$\mathrm{A} = \left[\begin{array}{cc} \alpha_1 & 0 \\ 1 & \alpha_2 \end{array} \right]. \tag{2.23}$$

One interesting thing to note is that the angle of the skew in the nodal structure can be found by finding the angle of the eigenvectors of this matrix.

Spirals The final type of node is the spiral. Spirals occur when the eigenvalues of state matrix are complex in form, which will cause an underdamped response. This behavior is very similar to the center structure described above; however since the system is damped, all trajectories will tend to zero. This creates a spiral structure, as is shown in figure 2.4.

Special Case: Substable Manifolds Substable manifolds are a very rare situation found when one of the eigenvectors of a system is zero. In this case all of the any point on one of the phase axes is stable, creating an infinite number of equilibrium points. Depending on the sign of the other eigenvalue, the system can be stable or unstable.

Phase Portraits of Nonlinear Systems

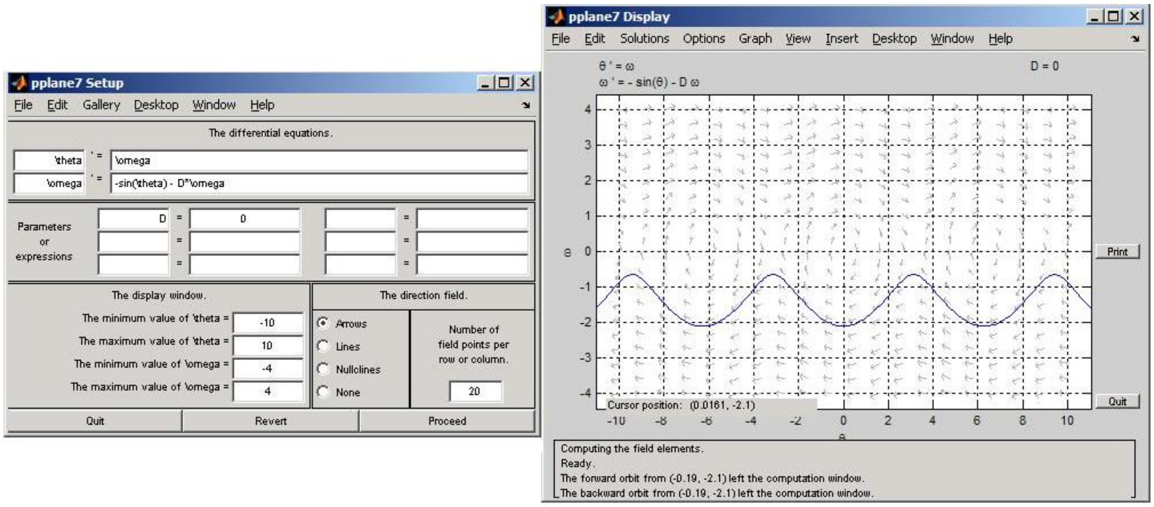
Phase portraits are a very important tool for understanding nonlinear system dynamics. Recalling from the previous chapter, one of the major differences between linear and nonlinear systems is that they have multiple equilibrium points. We may approximate the phase portrait near each of a nonlinear system's equilibrium points by solving the Jacobian matrix and analyzing the eigenstructure at each equilibrium point. Knowing the phase plane structure around these equilibrium points, we can sketch the phase portrait by drawing the appropriate dynamic around each equilibrium point. An example of the procedure of drawing a phase portrait for a non-linear system is shown below.

System from Example 1.10 Recall that the system in example 10 was described by the following differential equations:

$$\begin{aligned} \dot{x} &= (x-1)(y-1) \qquad (2.24) \\ \dot{y} &= x - y + 3, \qquad (2.25) \end{aligned}$$

with equilibria at $\vec{x} = [1 4]^T$, $[-2 1]$. The Jacobian matrix for this system of equations is:

$$\mathrm{J}(\vec{x}) = \left[\begin{array}{cc} y-1 & x-1 \\ 1 & -1 \end{array} \right]. \tag{2.26}$$


pplane7 Setup
File Edit Gallery Desktop Window Help
Quit
Revert
Proceed
pplane7 Display
File Edit Solutions Options Graph View Insert Desktop Window Help
Print
Quit
4
3
2
1
0
-1
-2
-3
-4
ω
-10
-8
-6
-4
-2
0
2
4
6
8
10
θ

The differential equations.

\theta ' = \omega

\omega ' = -sin(\theta) - D*\omega

Parameters or expressions: D = 0

The display window.

The minimum value of \theta = -10

The maximum value of \theta = 10

The minimum value of \omega = -4

The maximum value of \omega = 4

The direction field.

● Arrows

○ Lines

○ Nullclines

○ None

Number of field points per row or column. 20

θ ' = ω

ω ' = - sin(θ) - D ω

D = 0

Cursor position: (0.0161, -2.1)

```
Computing the field elements.
Ready.
The forward orbit from (-0.19, -2.1) left the computation window.
The backward orbit from (-0.19, -2.1) left the computation window.
```

Figure 2.5: Left, the pplane7 interface; right, the resulting phase portrait from the nonlinear differential equation entered into the pplane7 interface.

Evaluating this equation at the $\vec{x_{eq,1}} = [1 \;\; 4]^T$ and $\vec{x_{eq,2}} = [-2 \;\; 1]$, we find the eigenvalues to be $\lambda = -1.23, 3.23$ and $\lambda = -.5 \pm 1.66j$ for the two equilibria, respectively. Therefore, at $\vec{x_{eq,1}} = [1 \;\; 4]^T$ there will be a saddle point since the eigenstructure at this equilibrium point has one positive and one negative value, and at $\vec{x_{eq,2}} = [-2 \;\; 1]$ there will be a spiral sink since the eigenstructure at this equilibrium point has two imaginary eigenvalues with negative real components. The actual phase portrait for this system is shown in figure 2.22.

Drawing Phase Portraits using pplane7

pplane7 is a matlab tool which will plot linear and nonlinear phase portraits for you given a 2^{nd} order set of differential equations. The files necessary to run pplane7 (pplane7.m and dfield7.m) can be found at http://math.rice.edu/ dfield/. To run pplane7, simply save these files to the local matlab directory and run the command `pplane7` to activate the graphical user interface, which is shown in figure 2.5 on the left. Once the interface is activated, a second order set of differential equations (linear or nonlinear) may be placed into the x' and y' fields, and any constant parameters the user may want to change may be placed in the 'parameters or expressions' fields. Finally the bounds of the plot may be set using the fields on the bottom left of the menu. To plot or update the phase portrait, simply click proceed. An example of a resulting phase portrait is shown on the right of figure 2.5. To create a trajectory, simply click on the phase portrait at a point you which for the trajectory to intersect, and pplane7 will calculate the trajectory from $t \in (-\infty, \infty)$.

2.2 The Canonical 2^{nd} Order System

To begin to understand the forced dynamics of LTI systems, we must first define the system in terms of the input-output properties of the system. We will begin by writing a general transfer function which describes most second order systems, and use this transfer function to establish a variety of methods to describe the output dynamics of the system based on the parameters of the transfer function.

2.2.1 The Generic 2^{nd} Order Transfer Function

The transfer function of a general second order LTI system is often represented as:

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{2.27}$$

where we define ω_n as the natural frequency of the system but, as will be demonstrated below, the natural frequency of the system is *not* the frequency at which the system oscillates. ζ is defined as the damping ratio, which is simply the ratio:

$$\zeta = \frac{\text{Exponential Decay Frequency}}{\omega_n}$$

.

Given this transfer function, we can calculate the pole locations for this system.

$$s_{pole} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{2.28}$$

We often assume the system will have underdamped behavior because when we begin to design control systems, underdamped behavior is often required because it allows us to get our system response near the desired set point faster than a critically or underdamped system. To produce an underdamped response, we must assume that ζ is less than 1, causing the square root term to be negative. Factoring out the negative term from the square root term, the poles are then represented as:

$$s_{pole} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}. \tag{2.29}$$

To simplify the above equation, we often define a new term:

$$\beta = \sqrt{1 - \zeta^2}. \tag{2.30}$$

Thus our pole position expression changes to:

$$s_{pole} = -\zeta\omega_n \pm j\beta\omega_n. \tag{2.31}$$

Therefore, as β approaches 1, the system will approach critically damped behavior and as β approaches 0, the system will approach undamped behavior.

2.2.2 The Step Response and its Performance Metrics

When designing control systems, we are often most concerned with the system response to a step input. The reason for this is two fold. First, controlled systems often are exhibited to discrete changes in their set point. For example, when changing the temperature set point on your stove, I would guess that you don't slowly increase the temperature by slowly rotating the set point knob. Instead you quickly change the temperature from one point to another. Second, a step change in input to a system is a very severe disturbance to that system. Therefore, in a sense, the step response is a way to analyze the dynamics of a system under 'worst case scenario' conditions.

To start, using the transfer function above, we can find the Laplace domain step response of the system:

$$Y(s) = T(s)U(s) = \left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)\frac{1}{s}. \tag{2.32}$$

Taking the inverse Laplace transform with initial conditions $y(0) = y'(0) = 0$, we find that the time domain solution to our generic system with a step input is of the form:

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\frac{\zeta}{\beta}sin(\omega_n\beta t) + cos(\omega_n\beta t)\right). \tag{2.33}$$

This solution is shown graphically in figure 2.6. This graph is labeled with many of the physical parameters we wish to use to describe our system, which are described below.

Peak Time (Tp)

The peak time parameter (often abbreviated Tp) is defined as the amount of time the system takes to reach the systems maximum value. In other words, we would like to find the time at which our system response reaches its global maximum, which for a stable system is the first local maximum as well. Thus, to find the local maxima of the system, we must find the first zero of the derivative of equation 2.33.

Taking the derivative and simplifying, we find:

$$\dot{y} = \frac{w_n}{\beta}e^{-\zeta\omega_n t}sin(\beta\omega_n t). \tag{2.34}$$

Setting this equation to zero, we find that:

$$\begin{aligned} 0 &= \frac{w_n}{\beta}e^{-\zeta\omega_n t}sin(\beta\omega_n t) && (2.35)\\ 0 &= e^{-\zeta\omega_n t}sin(\beta\omega_n t). && (2.36)\end{aligned}$$

since the exponential term will only reach zero at $t \rightarrow \infty$, we may eliminate it from the equation as well. Therefore,

$$\begin{aligned} 0 &= sin(\beta\omega_n t) && (2.37) \\ \text{or, } sin^{-1}(0) &= \beta\omega_n t. && (2.38) \end{aligned}$$

Since the sine function is zero for all values $\theta = n\pi$, the above expression becomes:

$$n\pi = \beta\omega_n t. \tag{2.39}$$

This can be further simplified, recalling that we are concerned with the first maximum of the system (i.e., when $n = 1$):

$$\pi = \beta\omega_n t. \tag{2.40}$$

Solving for the time, t, we find that the peak time can be defined as:

$$t = \text{Tp} = \frac{\pi}{\beta\omega_n}. \tag{2.41}$$

Peak Response (Mp)

Now that we have defined the time at which the the response is at its maximum, we can define the maximum value which the system response will reach in terms of the

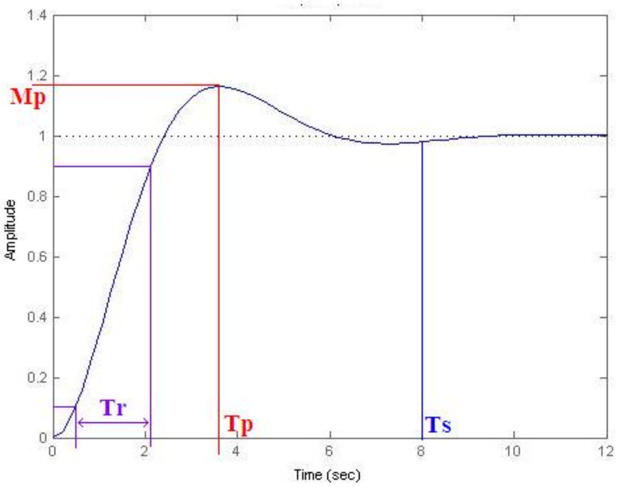


Figure 2.6: Step response of an underdamped second order LTI system and the location of the many physical parameters we will describe mathematically.

peak time. This maximum is often called the peak response, which is often denoted as Mp. Thus, the maximum value of the system response is:

$$\mathrm{Mp} = 1 - e^{-\zeta\omega_n \,\mathrm{Tp}}\left(\frac{\zeta}{\beta} sin(\omega_n\beta\,\mathrm{Tp}) + cos(\omega_n\beta\,\mathrm{Tp})\right). \tag{2.42}$$

Substituting equation 2.41 into the above equation, we find:

$$\begin{aligned}\mathrm{Mp} &= 1 - e^{-\zeta\omega_n\pi/(\beta\omega_n)}\left(\frac{\zeta}{\beta} sin(\omega_n\beta\pi/(\beta\omega_n)) + cos(\omega_n\beta\pi/(\beta\omega_n))\right) && (2.43)\\ \mathrm{Mp} &= 1 - e^{-\zeta\pi/\beta}\left(\frac{\zeta}{\beta} sin(\pi) + cos(\pi)\right). && (2.44)\end{aligned}$$

Simplifying further, we find that the definition of the peak response is:

$$\mathrm{Mp} = 1 - e^{-\zeta\pi/\beta}. \tag{2.45}$$

Percent Overshoot (PO)

The percent overshoot is simply a redefinition of the peak response. This parameter is used more often by control engineers as it is true for all possible step inputs, as opposed to the assumption that the peak response metric utilizes in that it is only true for the unit step impulse. Thus, we define the percent overshoot as:

$$\mathrm{PO} = (\,\mathrm{Mp} - 1) \times 100 = 100e^{\zeta\pi/\beta}. \tag{2.46}$$

Settling Time (Ts)

The settling time of the response describes how quickly the dynamics of the system reach a steady state. The settling time is (somewhat arbitrarily) defined as the amount of time it takes for the envelop of the system response to remain within ±2% of its steady state value. Therefore, starting with the original step response equation,

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\frac{\zeta}{\beta} sin(\omega_n\beta t) + cos(\omega_n\beta t)\right), \tag{2.47}$$

the envelop of the response (i.e., the exponential term of step response) will reach 2% when:

$$\begin{aligned}0.02 &= e^{-\zeta\omega_n t} && (2.48)\\ -ln(0.02) &= \zeta\omega_n t. && (2.49)\end{aligned}$$

Solving for t, we find that:

$$t = \frac{-ln(0.02)}{\zeta\omega_n}, \tag{2.50}$$

Simplifying, we find the definition of settling time to be:

$$\text{Ts} = \frac{4}{\zeta\omega_n}. \tag{2.51}$$

Rise Time (Tr)

The rise time is another quantitative measure of the system response to a step input. The rise time is defined as the amount of time the initial rise of the system response to go from 10 percent to 90 percent of its steady state value. Although there is no mathematical construct to specifically define the rise time, it is still a very useful measure of how rapidly a system can come near to its steady state.

Steady State Error (SSE)

When analyzing the step response of linear systems there is not always a one-to-one relationship between the input and output of the system. This results in a unit step response which will have a steady state which is not 1. This is shown in figure 2.7. Although for a normal system measuring the steady state value is of the system is only of minor importance, when dealing with control systems the divergence of the controlled system response from its desired value is of paramount importance.

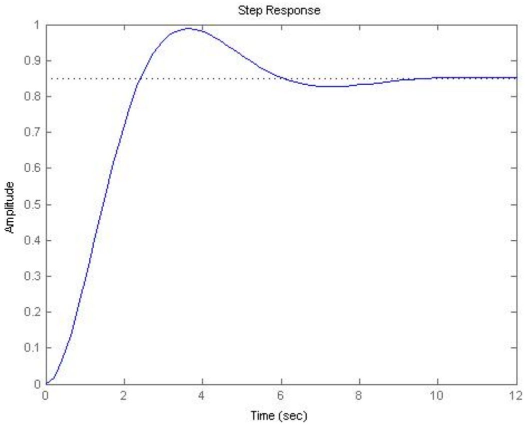


Figure 2.7: Step response of a system with a non-zero steady state error.

The steady state value of a system can be found using the final value theorem. This theorem states that the final value of the system response can be found by the following equation:

$$y(\infty) = \lim_{s \to \infty} sT(s)U(s), \tag{2.52}$$

where s is the Laplace variable, $T(s)$ is the transfer function of the system and $U(s)$ is the Laplace domain transfer function of the input function.

Expanding on this concept is the measurement of steady state error of the unit step response. The steady state error is defined as:

$$\text{sse} = \text{ExpectedSteadyState} - \text{SystemSteadyState}.$$

Thus, the steady state error for the step response (i.e., when $U(s) = 1/s$) can be calculated as:

$$\text{sse} = 1 - \lim_{s \to \infty} sT(s)\frac{1}{s} = 1 - \lim_{s \to \infty} T(s) \tag{2.53}$$

for the unit step response.

2.3 Supplemental: Explicitly Solving Sets of Linear Differential Equations

Coming soon!

2.3.1 The Matrix Exponential

2.3.2 Linear Transformation: Diagonalization

2.3.3 Solution using Matrix Exponential Method

Chapter 3

Control Theory

3.1 Laplace Domain Controllers

3.1.1 Feedback and the Error Function

3.1.2 Proportional Control

3.1.3 Proportional-Integral Control: Steady State Error Elimination

3.1.4 Proportional-Derivitive Control: Damping

3.1.5 PID Control

3.2 Design via the Root Locus

3.2.1 Theoretical Construct

3.2.2 Sketching the Root Locus

3.2.3 Using Computer Tools for Controller Design

3.3 Analysis of the Control System

3.3.1 Sensitivity and Robust Control

3.3.2 Disturbances

3.3.3 Sensor Noise