ME 230 Kinematics and Dynamics

Wei-Chih Wang
Department of Mechanical Engineering
University of Washington

Kinetics of a particle: Impulse and Momentum

Chapter 15

Chapter objectives

- Develop the principle of linear impulse and momentum for a particle
- Study the conservation of linear momentum for particles
- Analyze the mechanics of impact
- Introduce the concept of angular impulse and momentum
- Solve problems involving steady fluid streams and propulsion with variable mass

Figure: 15_COC
The design of the bumper cars used for this amusement park ride requires knowledge of the principles of impulse and momentum.

Lecture 10

- **Kinetics of a particle:**

Impulse and Momentum (Chapter 15)

- 15.1-15.3

Material covered

• **Kinetics of a particle: Impulse and Momentum**

- Principle of linear impulse and momentum

- Principle of linear impulse and momentum for a system of particles

- Conservation of linear momentum for a system of particles

…Next lecture…Impact

Today's Objectives

<u>Students should be able to:</u>

- Calculate the linear momentum of a particle and linear impulse of a force
- Apply the principle of linear impulse and momentum
- Apply the principle of linear impulse and momentum to a system of particles
- Understand the conditions for conservation of momentum

W. Wang

Applications 1

A dent in an automotive fender can be removed using an impulse tool, which delivers a force over a very short time interval.

How can we determine the magnitude of the linear impulse applied to the fender?

Could you analyze a carpenter's hammer striking a nail in the same fashion?

Applications 2

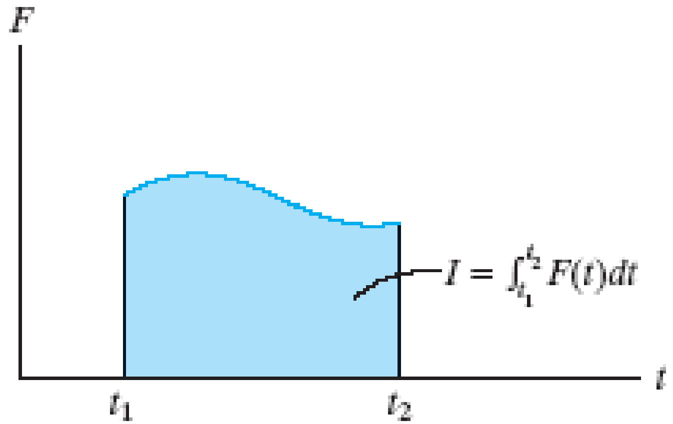
Sure! When a stake is struck by a sledgehammer, a large impulsive force is delivered to the stake and drives it into the ground.

If we know the initial speed of the sledgehammer and the duration of impact, how can we determine the magnitude of the impulsive force delivered to the stake?

Principle of linear impulse and momentum (Section 15.1)

Linear momentum: The vector m***v*** is called the linear momentum, denoted as ***L***. This vector has the same direction as ***v***. The linear momentum vector has units of (kg·m)/s or (slug·ft)/s. mass is constant!

Linear impulse: The integral ∫***F*** dt is the linear impulse, denoted ***I***. It is a vector quantity measuring the effect of a force during its time interval of action. ***I*** acts in the same direction as ***F*** and has units of N·s or lb·s.



W. Wang Variable Force

The impulse may be determined by direct integration. Graphically, it can be represented by the area under the force versus time curve. If ***F*** is constant, then

$$\boldsymbol{I} = \boldsymbol{F}\,(t_2 - t_1)\ .$$

Principle of linear impulse and momentum (continued) (Section 15.1)

The next method we will consider for solving particle kinetics problems is obtained by integrating the equation of motion with respect to time.

The result is referred to as the principle of impulse and momentum. It can be applied to problems involving both linear and angular motion.

This principle is useful for solving problems that involve force, velocity, and time. It can also be used to analyze the mechanics of impact (discussed in a later section).

Principle of linear impulse and momentum (continued) (Section 15.1)

The principle of linear impulse and momentum is obtained by integrating the equation of motion with respect to time. The equation of motion can be written

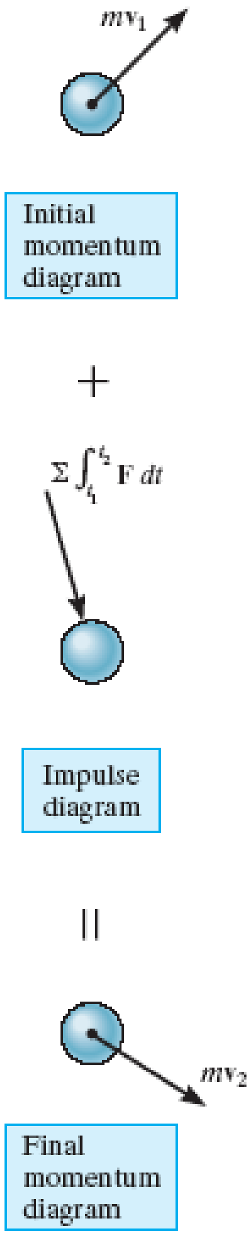
$$\sum \boldsymbol{F} = m\,\boldsymbol{a} = m\,(d\boldsymbol{v}/dt)$$

Separating variables and integrating between the limits $\boldsymbol{v} = \boldsymbol{v}_1$ at $t = t_1$ and $\boldsymbol{v} = \boldsymbol{v}_2$ at $t = t_2$ results in

$$\sum \int_{t_1}^{t_2} \boldsymbol{F}\,dt = m \int_{\boldsymbol{v}_1}^{\boldsymbol{v}_2} d\boldsymbol{v} = m\boldsymbol{v}_2 - m\boldsymbol{v}_1$$

This equation represents the principle of linear impulse and momentum. It relates the particle's final velocity, $\boldsymbol{v}_2$, and initial velocity ($\boldsymbol{v}_1$) and the forces acting on the particle as a function of time.

Principle of linear impulse and momentum (continued) (Section 15.1)



The principle of linear impulse and momentum in vector form is written as

$$m\boldsymbol{v}_1 + \sum \int_{t_1}^{t_2} \boldsymbol{F}\, dt = m\boldsymbol{v}_2$$

The particle's initial momentum plus the sum of all the impulses applied from t_1 to t_2 is equal to the particle's final momentum.

The two momentum diagrams indicate direction and magnitude of the particle's initial and final momentum, $m\boldsymbol{v}_1$ and $m\boldsymbol{v}_2$. The impulse diagram is similar to a free body diagram, but includes the time duration of the forces acting on the particle.

Impulse and momentum: Scalar equations

Since the principle of linear impulse and momentum is a vector equation, it can be resolved into its x, y, z component scalar equations:

$$m(v_x)_1 + \sum \int_{t_1}^{t_2} F_x \, dt = m(v_x)_2$$

$$m(v_y)_1 + \sum \int_{t_1}^{t_2} F_y \, dt = m(v_y)_2$$

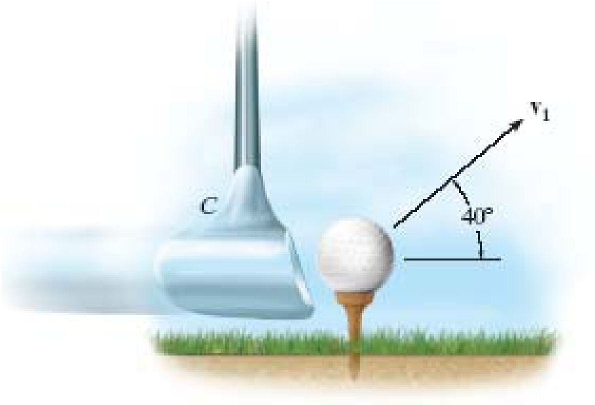
$$m(v_z)_1 + \sum \int_{t_1}^{t_2} F_z \, dt = m(v_z)_2$$

The scalar equations provide a convenient means for applying the principle of linear impulse and momentum once the velocity and force vectors have been resolved into x, y, z components

Problem solving

- Establish the x, y, z coordinate system.
- Draw the particle's free body diagram and establish the direction of the particle's initial and final velocities, drawing the impulse and momentum diagrams for the particle. Show the linear momenta and force impulse vectors.
- Resolve the force and velocity (or impulse and momentum) vectors into their x, y, z components, and apply the principle of linear impulse and momentum using its scalar form.
- Forces as functions of time must be integrated to obtain impulses. If a force is constant, its impulse is the product of the force's magnitude and time interval over which it acts.

Problem 1



Given: A 40 g golf ball is hit over a time interval of 3 ms by a driver. The ball leaves with a velocity of 35 m/s, at an angle of 40°. Neglect the ball's weight while it is struck.

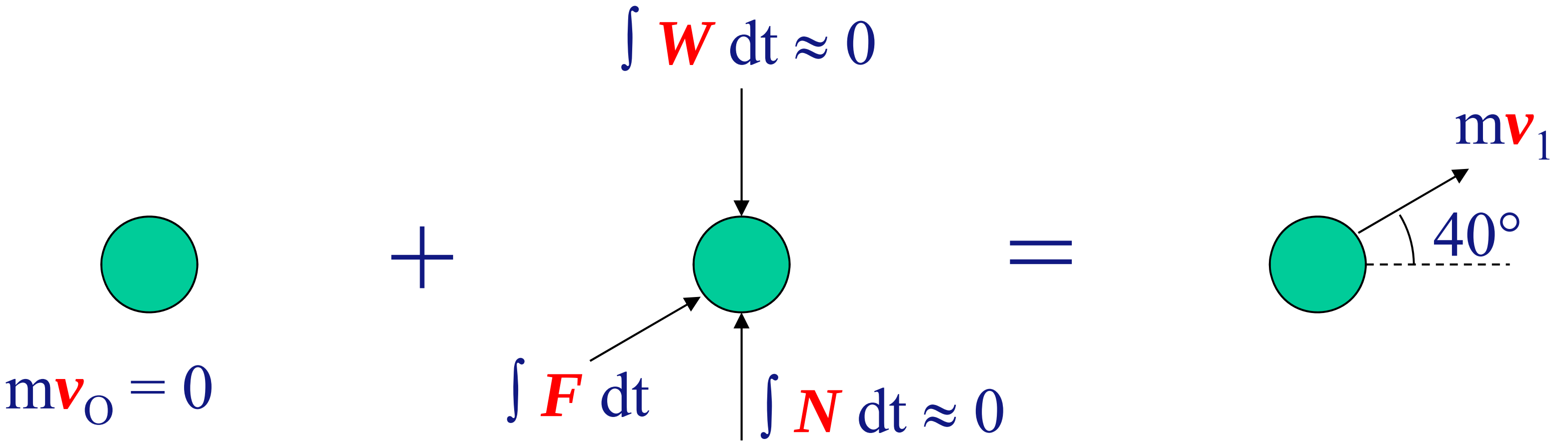
Find: The average impulsive force exerted on the ball and the momentum of the ball 1 s after it leaves the club face.

Plan:
1) Draw the momentum and impulsive diagrams of the ball as it is struck.
2) Apply the principle of impulse and momentum to determine the average impulsive force.
3) Use kinematic relations to determine the velocity of the ball after 1 s. Then calculate the linear momentum.

Problem 1 (continues)

Solution:

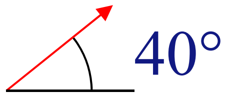
1) The impulse and momentum diagrams can be drawn:



The impulse caused by the ball's weight and the normal force N can be neglected because their magnitudes are very small as compared to the impulse of the club. Since the initial velocity (v_O) is zero, the impulse from the driver must be in the direction of the final velocity (v_1).

Problem 1 (continues)

2) The principle of impulse and momentum can be applied along the direction of motion:



$$mv_O + \sum \int_{t_0}^{t_1} F\,dt = mv_1$$

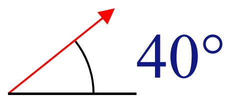
The average impulsive force can be treated as a constant value over the duration of impact. Using $v_O = 0$,

$$0 + \int_{0}^{0.003} F_{avg}\,dt = mv_1$$

$$F_{avg}(0.003 - 0) = mv_1$$

$$(0.003)\,F_{avg} = (0.04)(35)$$

$$F_{avg} = 467\text{ N}$$



Problem 1 (continues)

3) After impact, the ball acts as a projectile undergoing free-flight motion. Using the constant acceleration equations for projectile motion:

$$v_{2x} = v_{1x} = v_1 \cos 40° = 35 \cos 40° = 26.81 \text{ m/s}$$

$$v_{2y} = v_{1y} - gt = 35 \sin 40° - (9.81)(1) = 12.69 \text{ m/s}$$

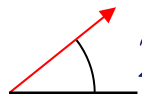
$$\Rightarrow \quad \boldsymbol{v}_2 = (26.81\,\boldsymbol{i} + 12.69\,\boldsymbol{j}) \text{ m/s}$$

The linear momentum is calculated as $\boldsymbol{L} = m\,\boldsymbol{v}$

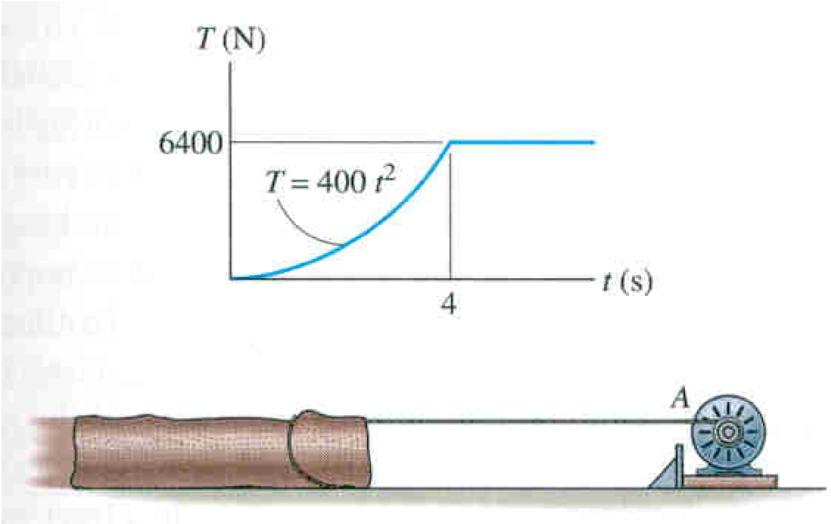
$$\boldsymbol{L}_2 = m\boldsymbol{v}_2 = (0.04)(26.81\,\boldsymbol{i} + 12.69\,\boldsymbol{j}) \text{ (kg·m)/s}$$

$$\boldsymbol{L}_2 = (1.07\,\boldsymbol{i} + 0.508\,\boldsymbol{j}) \text{ (kg·m)/s}$$

$$L_2 = 1.18 \text{ (kg·m)/s}$$


25.4°

Problem 2



Given: The 500 kg log rests on the ground (coefficients of static and kinetic friction are $\mu_s = 0.5$ and $\mu_k = 0.4$). The winch delivers a towing force ***T*** to its cable at A as shown.

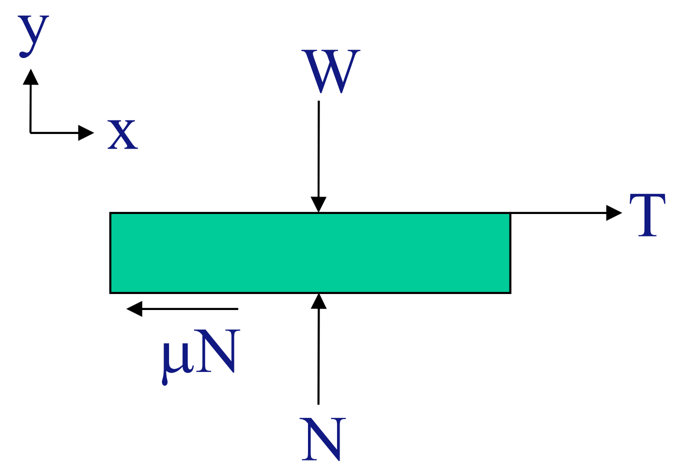
Find: The speed of the log when t = 5 s.

Plan:
1) Draw the FBD of the log.
2) Determine the force needed to begin moving the log, and the time to generate this force.
3) After the log starts moving, apply the principle of impulse and momentum to determine the speed of the log at t = 5 s.

Problem 2 (continues)

Solution:

1) Draw the FBD of the log:



$\sum F_y = 0$ leads to the result that $N = W = mg = (500)(9.81) = 4905$ N. Before the log starts moving, use μ_s. After the log is moving, use μ_k.

2) The log begins moving when the towing force T exceeds the friction force $\mu_s N$. Solve for the force, then the time.

$$\sum F_x = 0 \text{ -> } T = \mu_s N = (0.5)(4905) = 2452.5 \text{ N}$$

Given function

$$T = 400\, t^2 = 2452.5 \text{ N}$$

$$t = 2.476 \text{ s}$$

Since t < 4 s, the log starts moving before the towing force reaches its maximum value.

Problem 2 (continues)

3) Apply the principle of impulse and momentum in the x-direction from the time the log starts moving at $t_1 = 2.476$ s to $t_2 = 5$ s.

$$\xrightarrow{+} \quad mv_1 + \sum \int_{t_1}^{t_2} F\,dt = mv_2 \quad \text{where } v_1 = 0 \text{ at } t_1 = 2.476 \text{ s}$$

$$0 + \int_{2.476}^{5} T\,dt - \int_{2.476}^{5} \mu_k N\,dt = mv_2$$

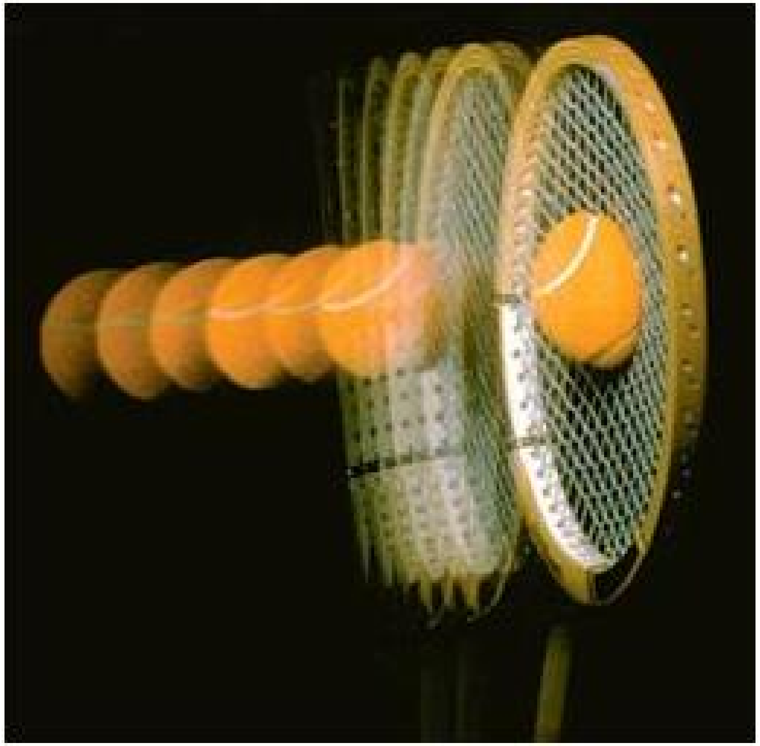
$$\int_{2.476}^{4} 400t^2\,dt + \int_{4}^{5} 6400\,dt - \int_{2.476}^{5} (0.4)(4905)\,dt = (500)v_2$$

$$(400/3)t^3 \Big|_{2.476}^{4} + (6400)(5 - 4) - (0.4)(4905)(5 - 2.476) = (500)v_2$$

$$\Rightarrow \quad v_2 = 15.9 \text{ m/s}$$

The kinetic coefficient of friction was used since the log is moving.

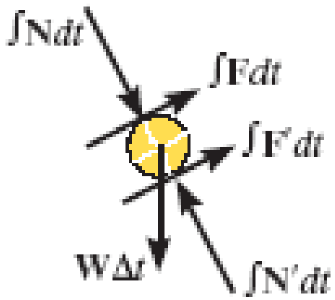
…we now continue with 15.2 and 15.3…



Applications 1

As the wheels of this pitching machine rotate, they apply frictional impulses to the ball, thereby giving it linear momentum in the direction of ***F***dt and ***F'***dt.

Does the release velocity of the ball depend on the mass of the ball?

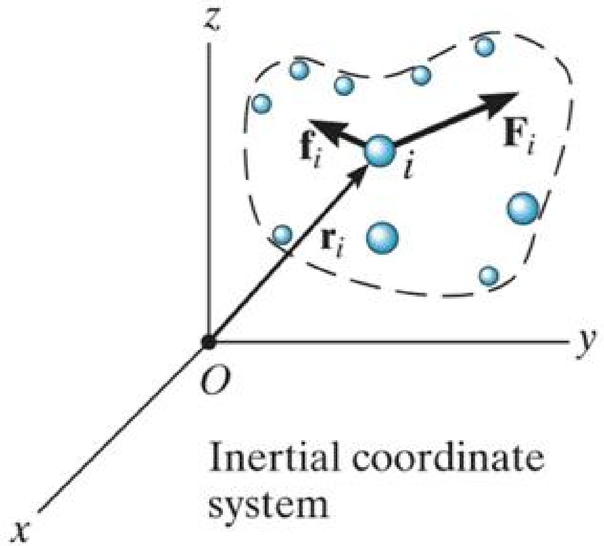


Applications 2

This large crane-mounted hammer is used to drive piles into the ground. Conservation of momentum can be used to find the velocity of the pile just after impact, assuming the hammer does not rebound off the pile.

If the hammer rebounds, does the pile velocity change from the case when the hammer doesn't rebound? Why?

Principle of linear impulse and momentum for a system of particles
(Section 15.2)

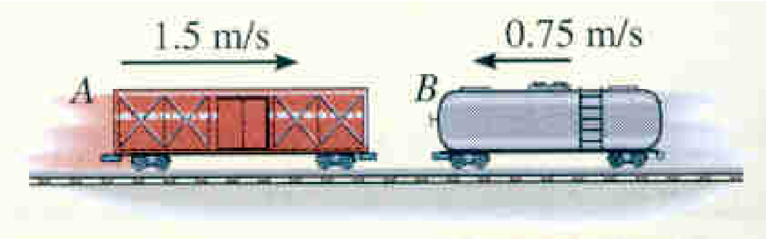


For the system of particles shown, the internal forces f_i between particles always occur in pairs with equal magnitude and opposite directions. Thus the internal impulses sum to zero.

The linear impulse and momentum equation for this system only includes the impulse of external forces.

$$\sum m_i(v_i)_1 + \sum \int_{t_1}^{t_2} F_i \, dt = \sum m_i(v_i)_2$$

Example



Given: Two rail cars with masses of m_A = 15 Mg and m_B = 12 Mg and velocities as shown.

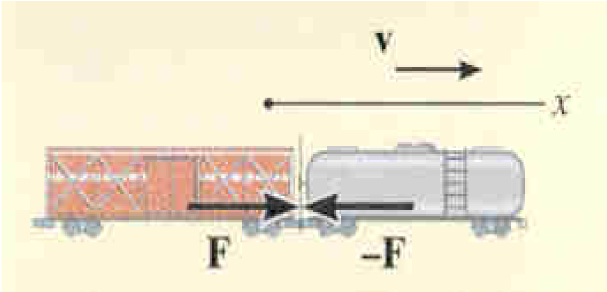
Find: The speed of the cars after they meet and connect. Also find the average impulsive force between the cars if the coupling takes place in 0.8 s.

Plan: Use conservation of linear momentum to find the velocity of the two cars after connection (all internal impulses cancel). Then use the principle of impulse and momentum to find the impulsive force by looking at only one car.

Example (continues)

Solution:

Conservation of linear momentum (x-dir):

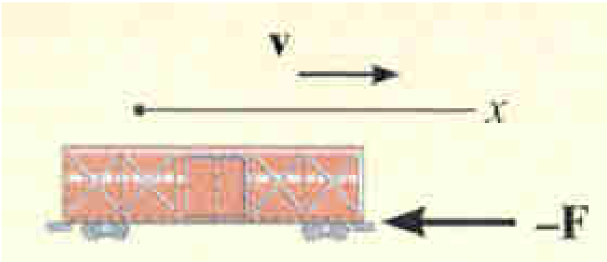


$$m_A(v_A)_1 + m_B(v_B)_1 = (m_A + m_B)\, v_2$$

$$15{,}000(1.5) + 12{,}000(-\, 0.75) = (27{,}000)v_2$$

$$v_2 = 0.5 \text{ m/s}$$

Impulse and momentum on car A (x-dir):



$$m_A(v_A)_1 + \int F\, dt = m_A(v_2)$$

$$15{,}000(1.5) - \int F\, dt = 15{,}000(0.5)$$

$$\int F\, dt = 15{,}000 \text{ N}\cdot\text{s}$$

The average force is

$$\int F\, dt = 15{,}000 \text{ N}\cdot\text{s} = F_{avg}(0.8 \text{ sec});\quad F_{avg} = 18{,}750 \text{ N}$$

Motion of the center of mass

For a system of particles, we can define a "fictitious" center of mass of an aggregate particle of mass m_{tot}, where m_{tot} is the sum ($\sum m_i$) of all the particles. This system of particles then has an aggregate velocity of $v_g = (\sum m_i v_i)/m_{tot}$.

The motion of this fictitious mass is based on motion of the center of mass for the system. The position vector $r_g = (\sum m_i r_i)/m_{tot}$ describes the motion of the center of mass.

Conservation of linear momentum for a system of particles
(Section 15.3)

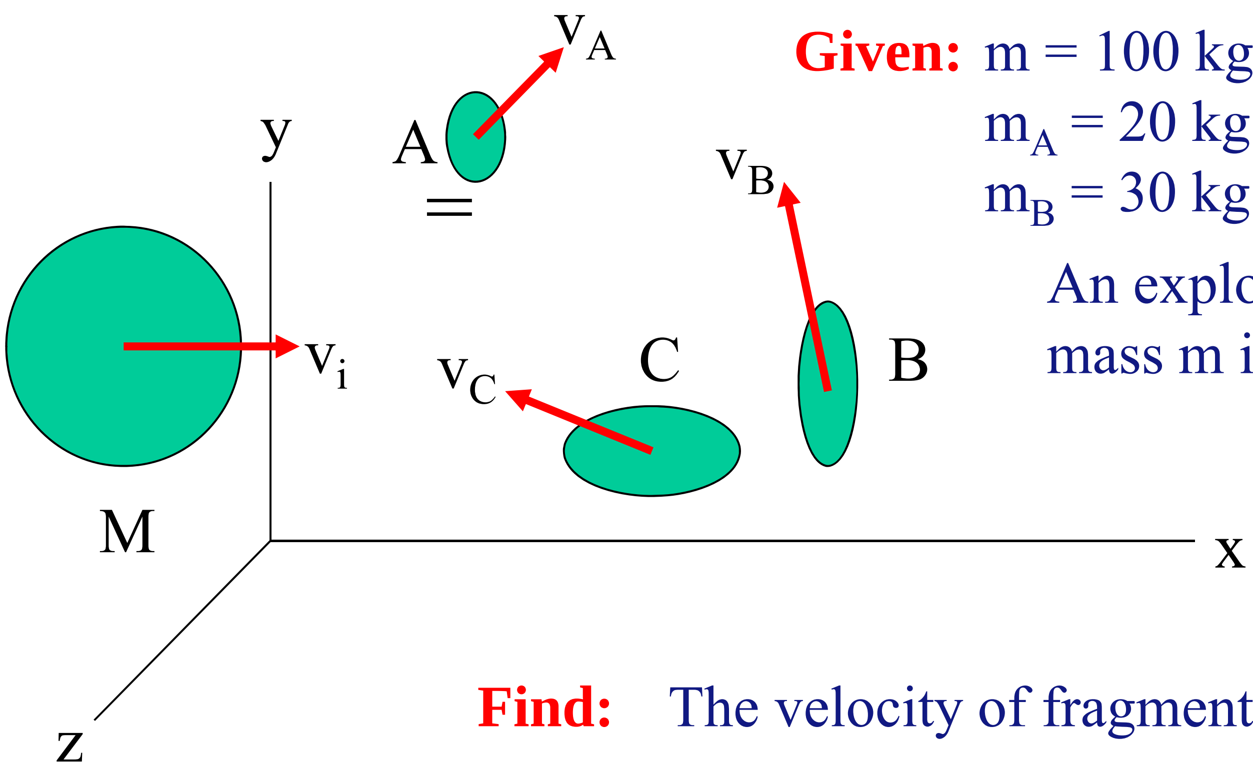
When the sum of external impulses acting on a system of objects is zero, the linear impulse-momentum equation simplifies to

$$\sum m_i(\boldsymbol{v}_i)_1 = \sum m_i(\boldsymbol{v}_i)_2$$

This important equation is referred to as the conservation of linear momentum. Conservation of linear momentum is often applied when particles collide or interact. When particles impact, only impulsive forces cause a change of linear momentum.

The sledgehammer applies an impulsive force to the stake. The weight of the stake can be considered negligible, or non-impulsive, as compared to the force of the sledgehammer. Also, provided the stake is driven into soft ground with little resistance, the impulse of the ground's reaction on the stake can also be considered negligible or non-impulsive.

Example 1



Given: m = 100 kg, $\boldsymbol{v}_i = 20\boldsymbol{j}$ (m/s)
$m_A = 20$ kg, $\boldsymbol{v}_A = 50\boldsymbol{i} + 50\boldsymbol{j}$ (m/s)
$m_B = 30$ kg, $\boldsymbol{v}_B = -30\boldsymbol{i} - 50\boldsymbol{k}$ (m/s)

An explosion has broken the mass m into 3 smaller particles.

Find: The velocity of fragment C after the explosion.

Plan: Since the internal forces of the explosion cancel out, we can apply the conservation of linear momentum to the SYSTEM.

Example 1 (continues)

Solution:

$$m\boldsymbol{v}_i = m_A\boldsymbol{v}_A + m_B\boldsymbol{v}_B + m_C\boldsymbol{v}_C$$

$$100(20\boldsymbol{j}) = 20(50\boldsymbol{i} + 50\boldsymbol{j}) + 30(-30\boldsymbol{i}-50\boldsymbol{k}) + 50(v_{cx}\,\boldsymbol{i} + v_{cy}\boldsymbol{j} + v_{cz}\,\boldsymbol{k})$$

Equating the components on the left and right side yields:

$0 = 1000 – 900 + 50(v_{cx})$ $\quad v_{cx} = -2$ m/s

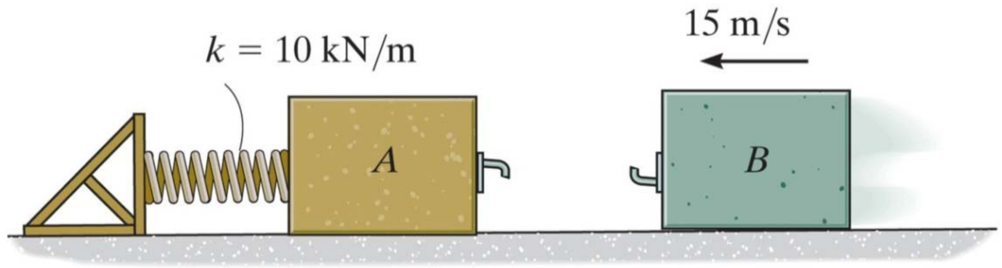
$2000 = 1000 + 50\ (v_{cy})$ $\quad v_{cy} = 20$ m/s

$0 = -1500 + 50\ (v_{cz})$ $\quad v_{cz} = 30$ m/s

So $\boldsymbol{v}_c = (-2\boldsymbol{i} + 20\boldsymbol{j} + 30\boldsymbol{k})$ m/s immediately after the explosion.

Examples combine momentum with energy and relative velocity equations for problem solving

EXAMPLE



Given: Spring constant k = 10 kN/m

$m_A = 15$ kg, $v_A = 0$ m/s, $m_B = 10$ kg, $v_B = 15$ m/s

The blocks couple together after impact.

Find: The maximum compression of the spring.

Plan: 1) We can consider both blocks as a single system and apply the conservation of linear momentum to find the velocity after impact, but before the spring compresses.

2) Then use the energy conservation to find the compression of the spring

EXAMPLE (continued)

Solution:

1) Conservation of linear momentum

$$+\rightarrow \quad \sum m_i(\boldsymbol{v}_i)_0 = \sum m_i(\boldsymbol{v}_i)_1$$

$$10\,(-15\boldsymbol{i}) = (15+10)\,(v\,\boldsymbol{i}\,)$$

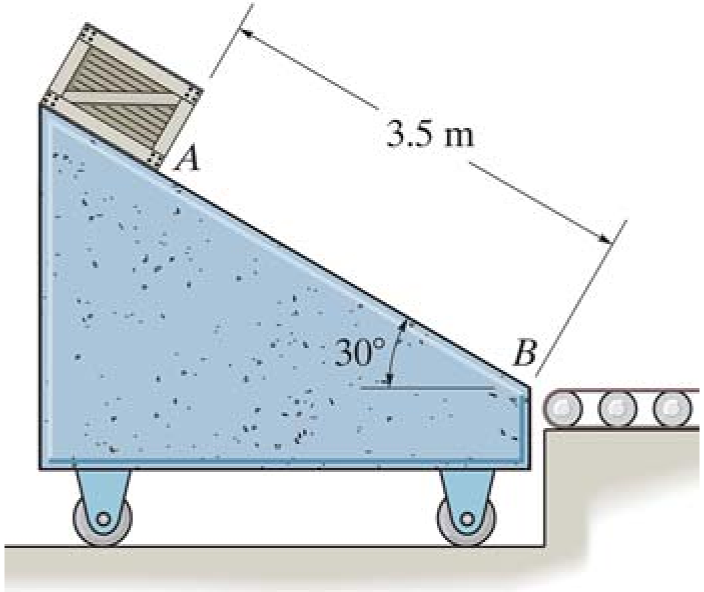
$$v = -6 \text{ m/s}$$
$$= 6 \text{ m/s} \leftarrow$$

2) Energy conservation equation

$$T_1 + V_1 = T_2 + V_2$$
$$0.5\,(15+10)\,(-6)^2 + 0 = 0 + 0.5\,(10000)\,x^2$$

So the maximum compression of the spring is $x = 0.3$ m.

Example



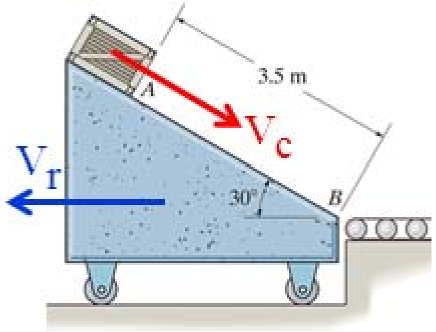
Given: The free-rolling ramp has a mass of 40 kg. The 10 kg crate slides from rest at A, 3.5 m down the ramp to B. Assume that the ramp is smooth, and neglect the mass of the wheels.

Find: The ramp's speed when the crate reaches *B*.

Plan: Use the energy conservation equation as well as conservation of linear momentum and the relative velocity equation (you really thought you could safely forget it?) to find the velocity of the ramp.

Example

Solution:



To find the relations between $\boldsymbol{v}_C$ and $\boldsymbol{v}_r$, use conservation of linear momentum:

$$\overset{+}{\rightarrow} \quad 0 = (40)(-v_r) + (10)\, v_{Cx}$$

$$\Rightarrow \quad v_{Cx} = 4\, v_r \qquad (1)$$

Since $\boldsymbol{v}_C = \boldsymbol{v}_r + \boldsymbol{v}_{C/r} \Rightarrow v_{Cx}\,\boldsymbol{i} - v_{Cy}\,\boldsymbol{j} = -v_r\,\boldsymbol{i} + v_{C/r}\,(\cos 30^o\,\boldsymbol{i} - \sin 30^o\,\boldsymbol{j})$

$$\Rightarrow \quad v_{Cx} = -\,v_r + v_{C/r}\cos 30^o \qquad (2)$$

$$v_{Cy} = v_{C/r}\sin 30^o \qquad (3)$$

Eliminating $v_{C/r}$ from Eqs. (2) and (3), and substituting Eq. (1) results in $v_{Cy} = 8.660\, v_r$

Example (continued)

Then, energy conservation equation can be written ;

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 10\,(9.81)(3.5\,\sin 30) = 0.5\,(10)(v_C)^2 + 0.5\,(40)(v_r)^2$$

$$\Rightarrow \quad 0 + 10\,(9.81)(3.5\,\sin 30)$$

$$= 0.5\,(10)\,[(4.0\,v_r)^2 + (8.660\,v_r)^2] + 0.5\,(40)\,(v_r)^2$$

$$\Rightarrow \quad 171.7 = 475.0\,(v_r)^2$$

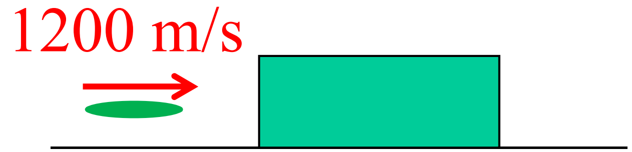
$$v_r = 0.601\ \text{m/s}$$

ATTENTION QUIZ

1. The 20 g bullet is fired horizontally at 1200 m/s into the 300 g block resting on a smooth surface. If the bullet becomes embedded in the block, what is the velocity of the block immediately after impact.

   A) 1125 m/s B) 80 m/s

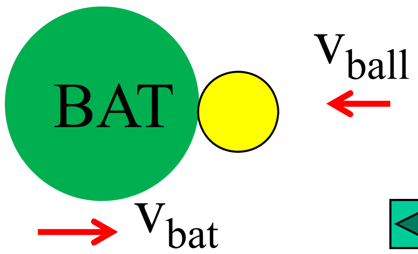
   C) 1200 m/s D) 75 m/s



2. The 200-g baseball has a horizontal velocity of 30 m/s when it is struck by the bat, B, weighing 900-g, moving at 47 m/s. During the impact with the bat, how many impulses of importance are used to find the final velocity of the ball?

   A) Zero B) One

   C) Two D) Three



Homework Assignment

Chapter15-6,11, 21,42, 54,57
Chapter15- 63, 80, 88, 92,105,108

Due next Wednesday !!!

Midterm Scores

- Mean: 64
- Median 63

Kinetics of a particle: Impulse and Momentum

Chapter 15

Chapter objectives

- Develop the principle of linear impulse and momentum for a particle
- Study the conservation of linear momentum for particles
- **Analyze the mechanics of impact**
- Introduce the concept of angular impulse and momentum
- Solve problems involving steady fluid streams and propulsion with variable mass

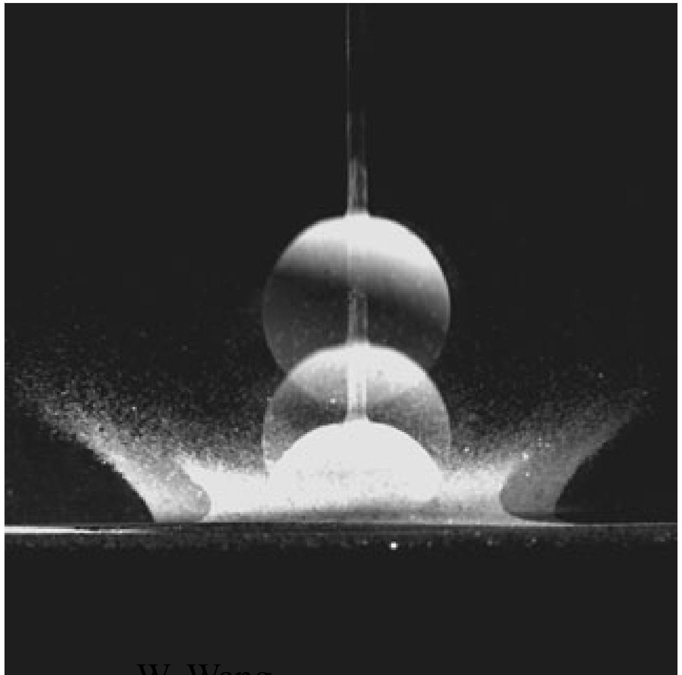
Figure: 15_COC
The design of the bumper cars used for this amusement park ride requires knowledge of the principles of impulse and momentum.

Lecture 11

- **Kinetics of a particle:**

Impact (Chapter 15)

- 15.4



Material covered

• **Kinetics of a particle:**

Impact

…Next lecture…

- Angular momentum

- Relation between moment of a force and angular momentum

- Angular impulse and momentum principles

NASA Deep Impact Project

The kinetic energy that will be released by the collision is estimated to be the equivalent of nearly 5 tons of TNT. 5 tons of TNT. However, this will only change the comet's velocity by about 0.0001 millimeters per second (0.014 inches per hour). The collision will not appreciably modify the orbital path of Tempel 1, which poses no threat to Earth now or in the foreseeable future.

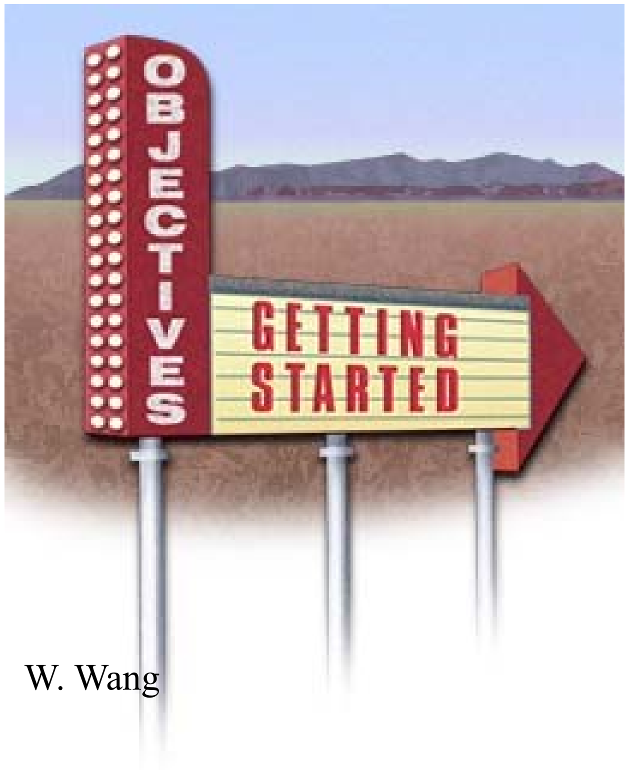
Also…

http://www.nasa.gov/mission_pages/deepimpact/multimedia/di-animation.html

Today's Objectives

<u>Students should be able to:</u>

1. Understand and analyze the mechanics of impact.
2. Analyze the motion of bodies undergoing a collision, in both central and oblique cases of impact.

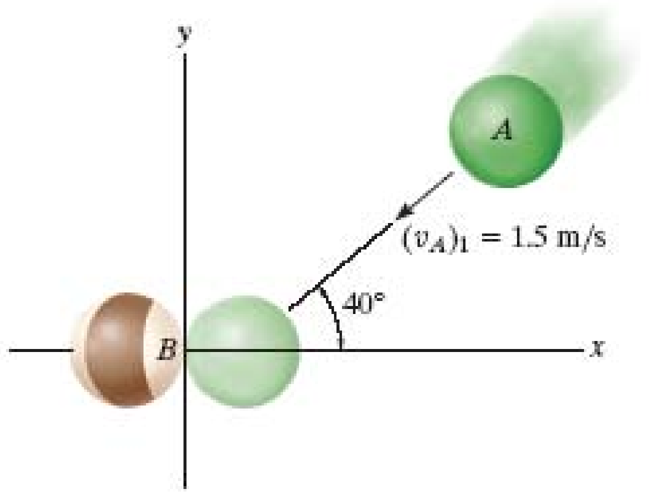


Applications 1

The quality of a tennis ball is measured by the height of its bounce. This can be quantified by the coefficient of restitution (restoration) of the ball.

If the height from which the ball is dropped and the height of its resulting bounce are known, how can we determine the coefficient of restitution of the ball?

Applications 2

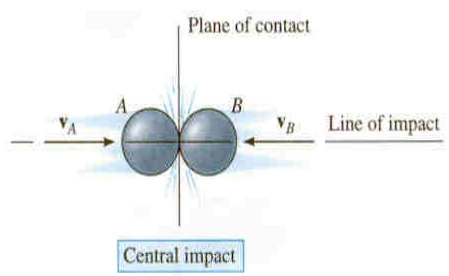


In a game of billiards, it is important to be able to predict the trajectory and speed of a ball after it is struck by another ball.

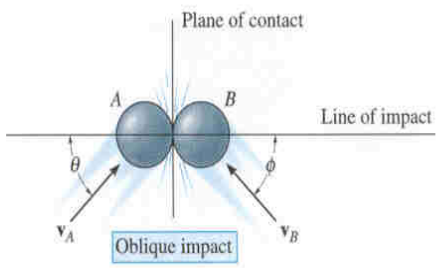
If we know the velocity of ball A before the impact, how can we determine the magnitude and direction of the velocity of ball B after the impact?

Impact (section 15.4)

Impact occurs when two bodies collide during a very short time period, causing large impulsive forces to be exerted between the bodies. Common examples of impact are a hammer striking a nail or a bat striking a ball. The line of impact is a line through the mass centers of the colliding particles. In general, there are two types of impact:



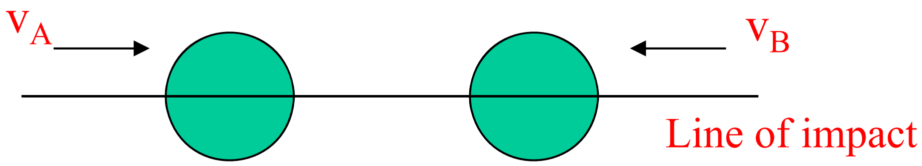
Central impact occurs when the directions of motion of the two colliding particles are along the line of impact.



Oblique impact occurs when the direction of motion of one or both of the particles is at an angle to the line of impact.

Central impact

Central impact happens when the velocities of the two objects are along the line of impact (recall that the line of impact is a line through the particles' mass centers).



Once the particles contact, they may deform if they are non-rigid. In any case, energy is transferred between the two particles.

There are two primary equations used when solving impact problems. The textbook provides extensive detail on their derivation.

Central impact (continued)

In most problems, the initial velocities of the particles, $(v_A)_1$ and $(v_B)_1$, are known, and it is necessary to determine the final velocities, $(v_A)_2$ and $(v_B)_2$. So the first equation used is the conservation of linear momentum, applied along the line of impact.

$$(m_A v_A)_1 + (m_B v_B)_1 = (m_A v_A)_2 + (m_B v_B)_2$$

This provides one equation, but there are usually two unknowns, $(v_A)_2$ and $(v_B)_2$. So another equation is needed. The principle of impulse and momentum is used to develop this equation, which involves the coefficient of restitution, or *e*.

Central impact (continued)

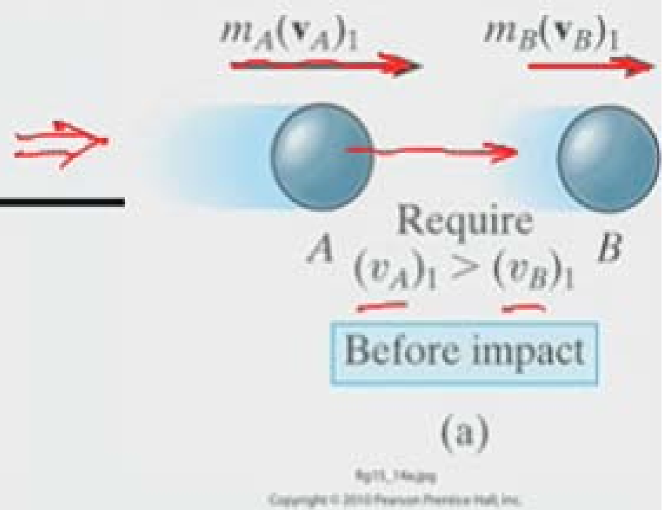
The coefficient of restitution, e, is the ratio of the particles' relative separation velocity after impact, $(v_B)_2 - (v_A)_2$, to the particles' relative approach velocity before impact, $(v_A)_1 - (v_B)_1$. The coefficient of restitution is also an indicator of the energy lost during the impact.

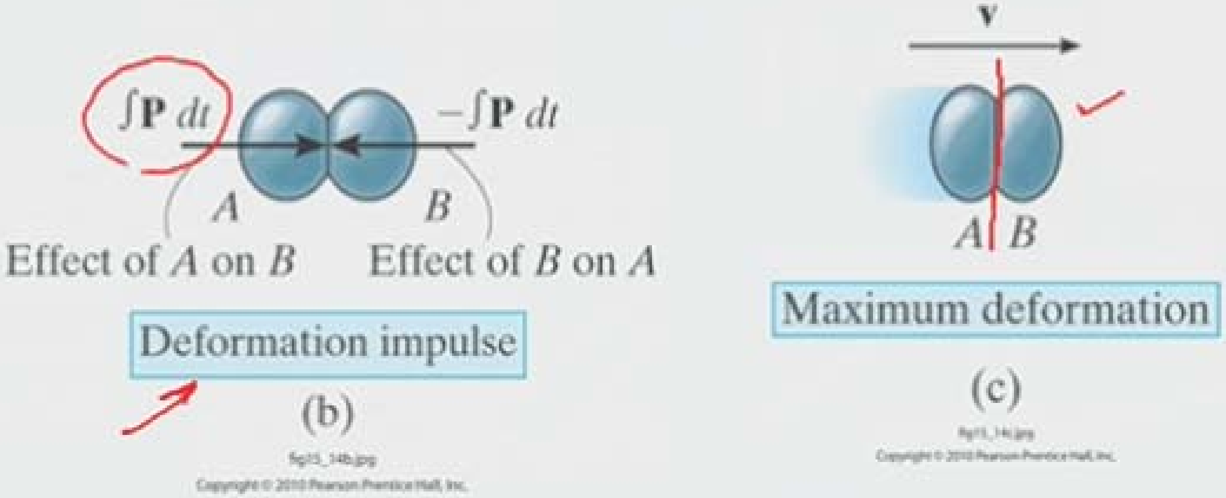
The equation defining the coefficient of restitution, e, is

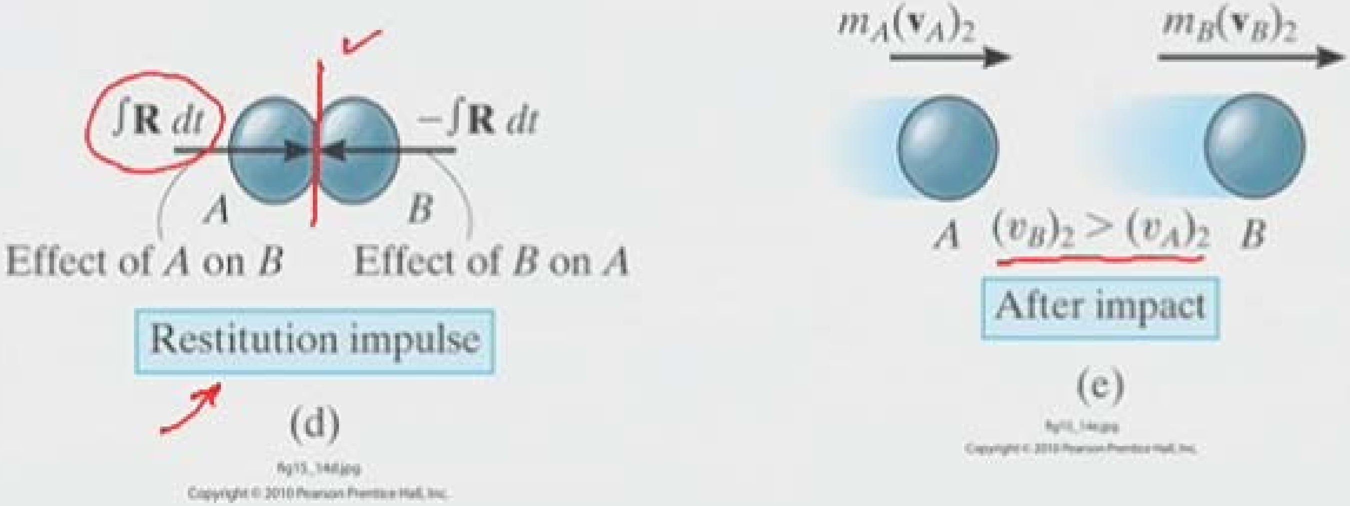
$$e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$$

If a value for e is specified, this relation provides the second equation necessary to solve for $(v_A)_2$ and $(v_B)_2$.

Concept: Impact (15.4)

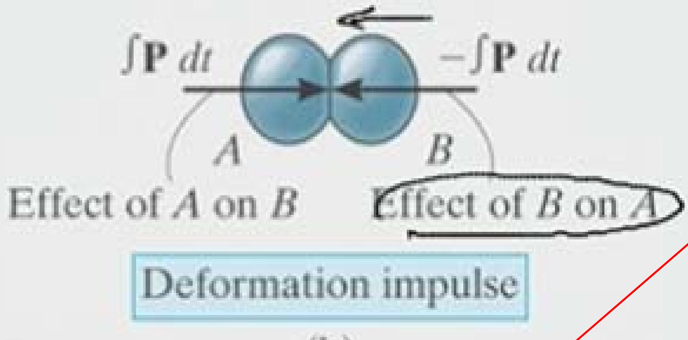






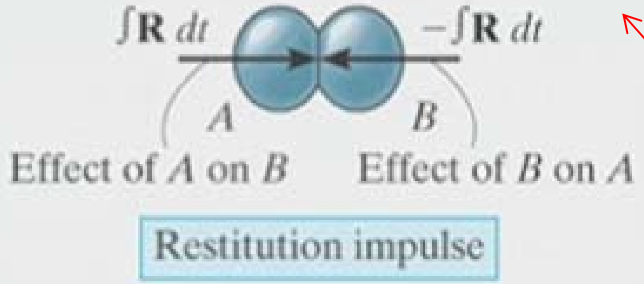
Theory: Restitution

system: momentum is conserved



(b)

$$\frac{\int Rdt}{\int Pdt} = e = \frac{(V_B)_2 - (V_A)_2}{(V_A)_1 - (V_B)_1}$$



(d)

$\int \bar{P}\,dt$: deformation

state 1) Pre- impact

@ max. def.

A: $m_A(\bar{v}_A)_1 - \int \bar{P}\,dt = m_A(v_A)$

$\int \bar{R}\,dt$: restitution

A: $m_A(v_A) - \int \bar{R}\,dt = m_A(v_A)_2$

Post- impact state 2

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Coefficient of restitution

In general, e has a value between zero and one. The two limiting conditions can be considered:

- Elastic impact ($e = 1$): In a perfectly elastic collision, no energy is lost and the relative separation velocity equals the relative approach velocity of the particles. In practical situations, this condition cannot be achieved.
- Plastic impact ($e = 0$): In a plastic impact, the relative separation velocity is zero. The particles stick together and move with a common velocity after the impact.

Some typical values of e are:

Steel on steel: 0.5 – 0.8

Wood on wood: 0.4 – 0.6

Lead on lead: 0.12 – 0.18

Glass on glass: 0.93 – 0.95

Impact: Energy losses

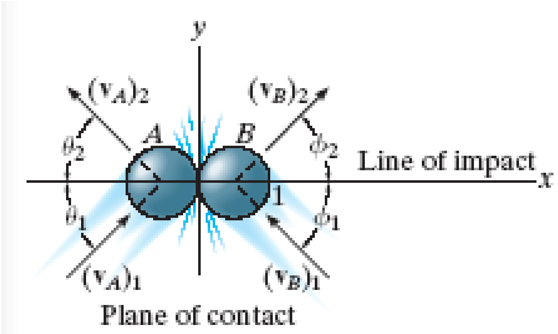
Once the particles' velocities before and after the collision have been determined, the energy loss during the collision can be calculated on the basis of the difference in the particles' kinetic energy. The energy loss is

$$\sum U_{1\text{-}2} = \sum T_2 - \sum T_1 \quad \text{where } T_i = 0.5 m_i (v_i)^2$$

During a collision, some of the particles' initial kinetic energy will be lost in the form of heat, sound, or due to localized deformation.

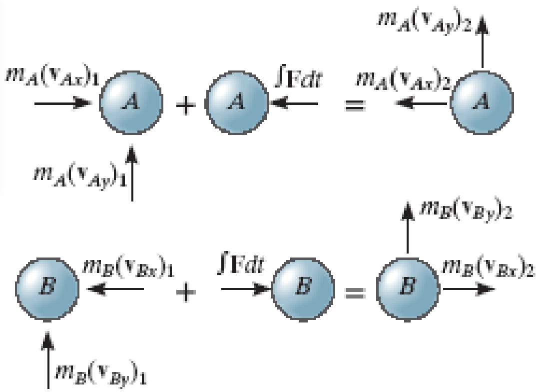
In a plastic collision ($e = 0$), the energy lost is a maximum, although it does not necessarily go to zero.

Oblique impact



In an oblique impact, one or both of the particles' motion is at an angle to the line of impact. Typically, there will be four unknowns: the magnitudes and directions of the final velocities.

The four equations required to solve for the unknowns are:



Conservation of momentum and the coefficient of restitution equation are applied along the line of impact (x-axis):

$$m_A(v_{Ax})_1 + m_B(v_{Bx})_1 = m_A(v_{Ax})_2 + m_B(v_{Bx})_2$$

$$e = [(v_{Bx})_2 - (v_{Ax})_2]/[(v_{Ax})_1 - (v_{Bx})_1]$$

Momentum of each particle is conserved in the direction perpendicular to the line of impact (y-axis):

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$$m_A(v_{Ay})_1 = m_A(v_{Ay})_2 \quad \text{and} \quad m_B(v_{By})_1 = m_B(v_{By})_2$$

Procedure of analysis

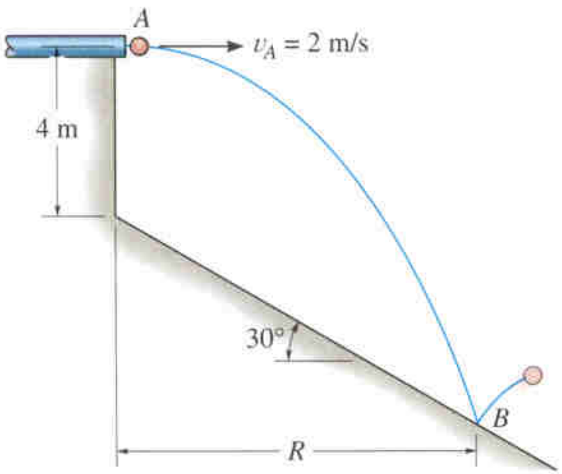
- In most impact problems, the initial velocities of the particles and the coefficient of restitution, *e*, are known, with the final velocities to be determined.

- Define the x-y axes. Typically, the x-axis is defined along the line of impact and the y-axis is in the plane of contact perpendicular to the x-axis.

- For both central and oblique impact problems, the following equations apply along the line of impact (x-dir.):
  $\sum m(v_x)_1 = \sum m(v_x)_2$ and $e = [(v_{Bx})_2 - (v_{Ax})_2]/[(v_{Ax})_1 - (v_{Bx})_1]$

- For oblique impact problems, the following equations are also required, applied perpendicular to the line of impact (y-dir.):
  $m_A(v_{Ay})_1 = m_A(v_{Ay})_2$ and $m_B(v_{By})_1 = m_B(v_{By})_2$

Example 1



Given: A 0.5 kg ball is ejected from the tube at A with a horizontal velocity v_A = 2 m/s. The coefficient of restitution at B is $e = 0.6$.

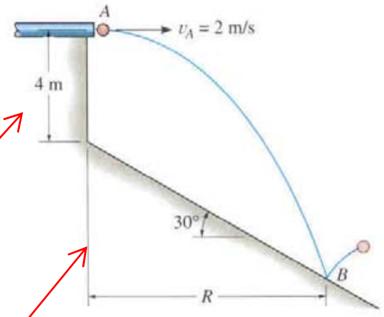
Find: The horizontal distance R where the ball strikes the smooth inclined plane and the speed at which it bounces from the plane.

Plan: 1) Use kinematics to find the distance R (projectile motion).

2) The collision at B is an oblique impact, with the line of impact perpendicular to the plane.

3) Thus, the coefficient of restitution applies perpendicular to the incline and the momentum of the ball is conserved along the incline.

Example 1 (continued)



Solution:

1) Apply the equations of projectile motion to determine R.
   Place the origin at A ($x_o = y_o = 0$) with the initial velocity of $v_{yo} = 0$, $v_{xo} = v_A = 2$ m/s:

   $x = x_o + v_{xo}t \quad \Rightarrow \quad R = 0 + 2t$

   $y = y_o + v_{yo}t - 0.5gt^2 \quad \Rightarrow \quad -(4 + R\tan 30°) = 0 + 0 - 0.5(9.81)t^2$

   Solving these equations simultaneously yields

   $$t = 1.028 \text{ s} \quad \text{and} \quad R = 2.06 \text{ m}$$

   It is also necessary to calculate the velocity of the ball just before impact:

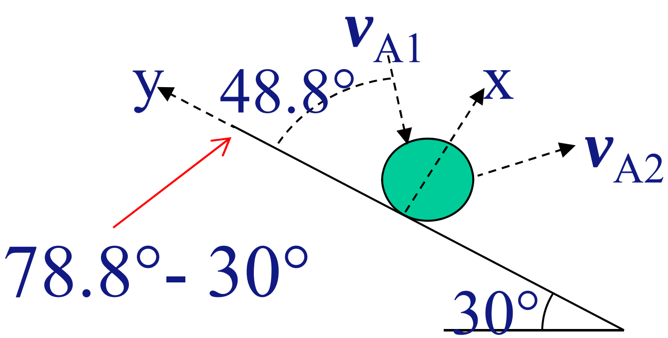
   $v_x = v_{xo} = 2$ m/s ($\rightarrow$)

   $v_y = v_{yo} - gt = 0 - 9.81(1.028) = -10.0886$ m/s ($\downarrow$)

   $\Rightarrow \quad v = 10.285$ m/s $\quad 78.8°$

Example 1 (continued)

2) Solve the impact problem by using x-y axes defined along and perpendicular to the line of impact, respectively:



Denoting the ball as A and plane as B, the momentum of the ball is conserved in the y-dir:

$$m_A(-v_{Ay})_1 = m_A(-v_{Ay})_2$$

$$(v_{Ay})_2 = (v_{Ay})_1 = v_A \cos 48.8° = 6.77 \text{ m/s}$$

The coefficient of restitution applies in the x-dir and $(v_{Bx})_1 = (v_{Bx})_2 = 0$ (assume incline doesn't move):

$$e = [(v_{Bx})_2 - (v_{Ax})_2]/[(v_{Ax})_1 - (v_{Bx})_1]$$

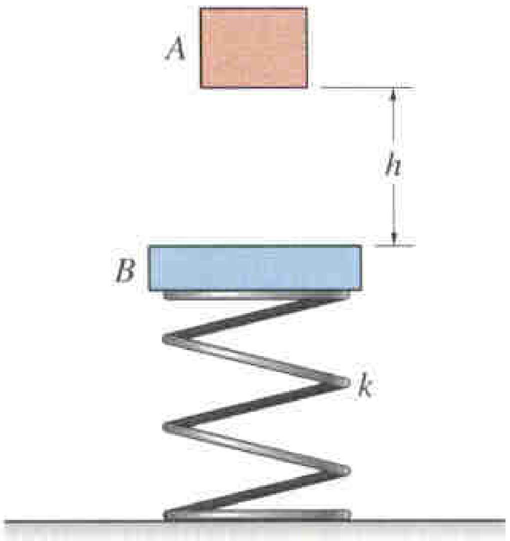
$$\Rightarrow \quad 0.6 = [0 - (v_{Ax})_2]/[-10.285 \sin 48.8° - 0]$$

$$\Rightarrow \quad (v_{Ax})_2 = 4.64 \text{ m/s}$$

The speed is the magnitude of the velocity vector:

$$v_{A2} = \sqrt{((v_{Ax})_2)^2 + ((v_{Ay})_2)^2} = 8.21 \text{ m/s}$$

Example 2



Given: A 2 kg block A is released from rest, falls a distance h = 0.5 m, and strikes plate B (3 kg mass). The coefficient of restitution between A and B is e = 0.6, and the spring stiffness is k = 30 N/m.

Find: The velocity of block A just after the collision.

Plan:
1) Determine the speed of the block just before the collision using kinematic equation or an energy method.
2) Analyze the collision as a central impact problem.

Example 2 (continued)

Solution:

1) Determine the speed of block A just before impact by using conservation of energy. Defining the gravitational datum at the initial position of the block ($h_1 = 0$) and noting the block is released from rest ($v_1 = 0$):

$$T_1 + V_1 = T_2 + V_2$$

$$0.5m(v_1)^2 + mgh_1 = 0.5m(v_2)^2 + mgh_2$$

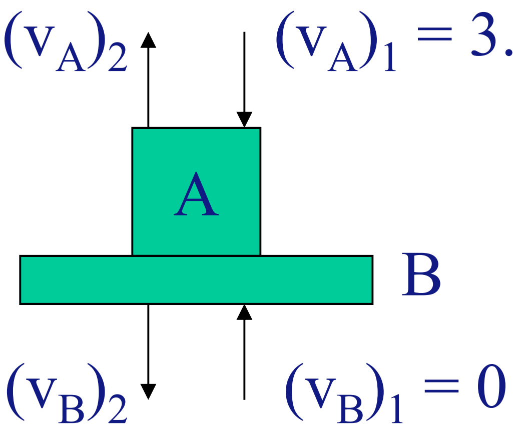
$$0 + 0 = 0.5(2)(v_2)^2 + (2)(9.81)(-0.5)$$

$$v_2 = 3.132 \text{ m/s} \downarrow$$

This is the speed of the block just before the collision. Plate (B) is at rest, velocity of zero, before the collision.

Example 2 (continued)

2) Analyze the collision as a central impact problem.



Apply conservation of momentum to the system in the vertical direction:

$+\uparrow \quad m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$

$(2)(-3.132) + 0 = (2)(v_A)_2 + (3)(v_B)_2$

Using the coefficient of restitution:

$+\uparrow \quad e = [(v_B)_2 - (v_A)_2]/[(v_A)_1 - (v_B)_1]$

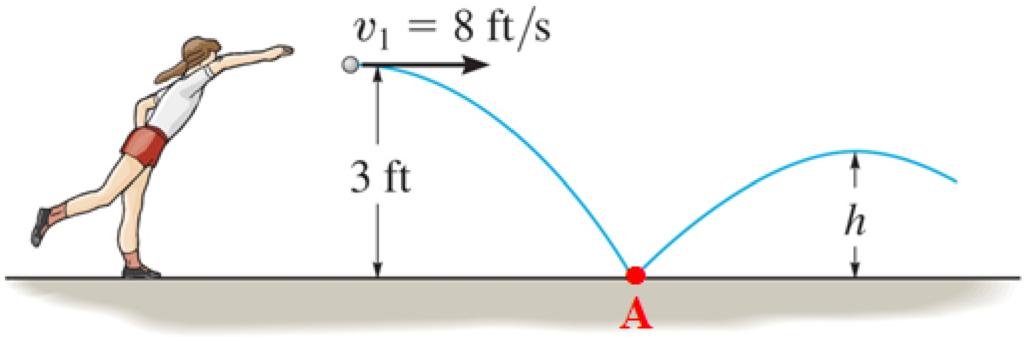
$\Rightarrow \quad 0.6 = [(v_B)_2 - (v_A)_2]/[-3.132 - 0] \quad \Rightarrow \quad -1.879 = (v_B)_2 - (v_A)_2$

Solving the two equations simultaneously yields

$(v_A)_2 = -0.125$ m/s $\downarrow$, $(v_B)_2 = -2.00$ m/s $\downarrow$

Both the block and plate will travel down after the collision.

Example 3



Given: The girl throws the ball with a velocity of v_1=8 ft/s. The coefficient of restitution between the ball and the hard ground is $e = 0.8$.

Find: The rebounding velocity of the ball at A, the maximum h.

Plan: 1) Determine the speed of the ball just before hitting the ground using projectile motion.
2) Apply the coefficient of restitution in the y-dir motion, and the conservation of momentum in the x-dir motion.
3) Use kinematics equations to find h

Example (continued)

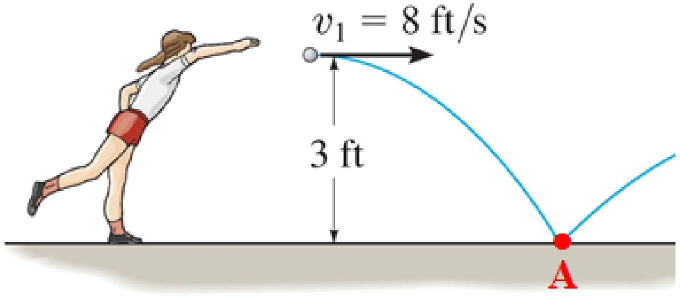
Solution:

1) By considering the vertical motion of the falling ball, we have:

$$(v_{Ay})^2 = (v_{1y})^2 + 2\, a_c\, (s_{Ay} - s_{1y})$$

$$(v_{Ay})^2 = 0 + 2\,(-32.2)\,(0 - 3)$$

$$v_{Ay} = -\,13.90 \text{ ft/s} = 13.90 \text{ ft/s} \downarrow$$



2) Apply the coefficient of restitution in the *y*-dir to determine the velocity of the ball just after it rebounds from the ground.

$$e = \frac{v_{g2} - (v_{Ay})_2}{(v_{Ay})_1 - v_{g1}} \quad \Rightarrow \quad 0.8 = \frac{0 - (v_{Ay})_2}{-13.90 - 0}$$

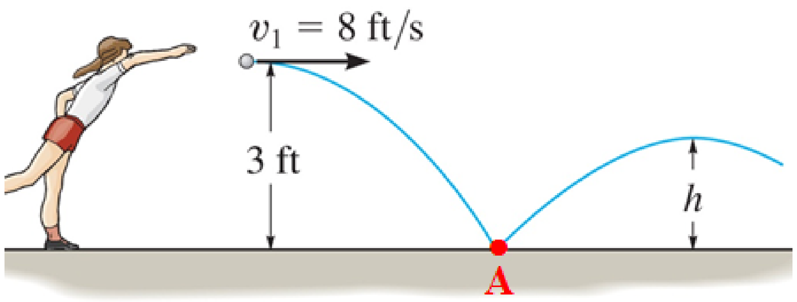
$$(v_{Ay})_2 = 11.12 \text{ ft/s} \uparrow$$

Example 3 (continued)

Apply conservation of momentum to the system in the x-dir :

$+\rightarrow$ $m\,(v_{1x}) = m\,(v_{Ax})$

$(v_{Ax}) = 8$ ft/s $\rightarrow$



Therefore, the rebounding velocity of the ball at A is

$(\boldsymbol{v}_A)_2 = (\,8\,\boldsymbol{i} + \;11.12\,\boldsymbol{j})$ ft/s

Result obtained from restitution equation from last page

3) Kinematics equations: the vertical motion of the ball after it rebounds from the ground is described by:

$(v_{2y})^2 = (v_{Ay})^2 + 2\,a_c\,(s_{2y} - s_{Ay})$ and v_{2y} is zero at height h

$0^2 = \;11.12^2 \;+ 2\,(-32.2)\,(h - 0)$

$h = 1.92$ ft

Homework Assignment

Chapter15-6,11, 21,42, 54,57
Chapter15- 63, 80, 88, 92,105,108

Due next Wednesday !!!

W. Wang

Kinetics of a particle: Impulse and Momentum

Chapter 15

Chapter objectives

- Develop the principle of linear impulse and momentum for a particle
- Study the conservation of linear momentum for particles
- Analyze the mechanics of impact
- Introduce the concept of angular impulse and momentum
- Solve problems involving steady fluid streams and propulsion with variable mass

Lecture 12

- **Kinetics of a particle:**

Angular momentum, relation between moment of a force and angular momentum, angular impulse and momentum principles (Chapter 15)

- 15.5-15.7

Material covered

- **Kinetics of a particle:**
- Angular momentum
- Relation between moment of a force and angular momentum
- Angular impulse and momentum principles

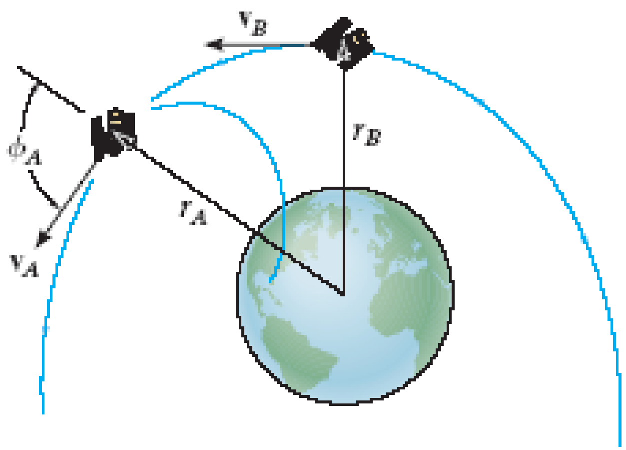
…Next lecture… start of Chapter 16

Today's Objectives

<u>Students should be able to:</u>

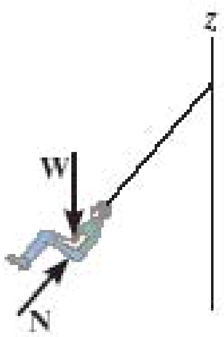
1. Determine the angular momentum of a particle and apply the principle of angular impulse & momentum.
2. Use conservation of angular momentum to solve problems.

Applications



Planets and most satellites move in elliptical orbits. This motion is caused by gravitational attraction forces. Since these forces act in pairs, the sum of the moments of the forces acting on the system will be zero. This means that angular momentum is conserved.

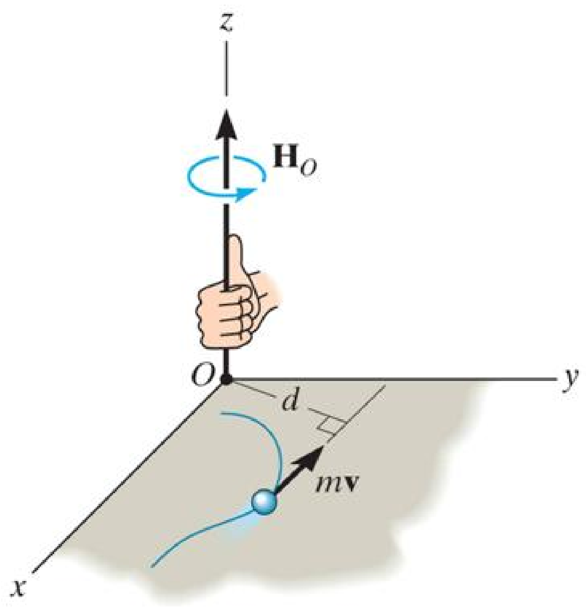
Applications (continued)



The passengers on the amusement-park ride experience conservation of angular momentum about the axis of rotation (the z-axis). As shown on the free body diagram, the line of action of the normal force, N, passes through the z-axis and the weight's line of action is parallel to it. Therefore, the sum of moments of these two forces about the z-axis is zero.

Angular momentum (15.5)

The angular momentum of a particle about point O is defined as the "moment" of the particle's linear momentum about O.



$$\boldsymbol{H}_o = \boldsymbol{r} \text{ x } m\boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ r_x & r_y & r_z \\ mv_x & mv_y & mv_z \end{vmatrix}$$

The magnitude of $\boldsymbol{H}_o$ is $(H_o)_z = mvd$

Relationship between moment of a force and angular momentum (15.6)

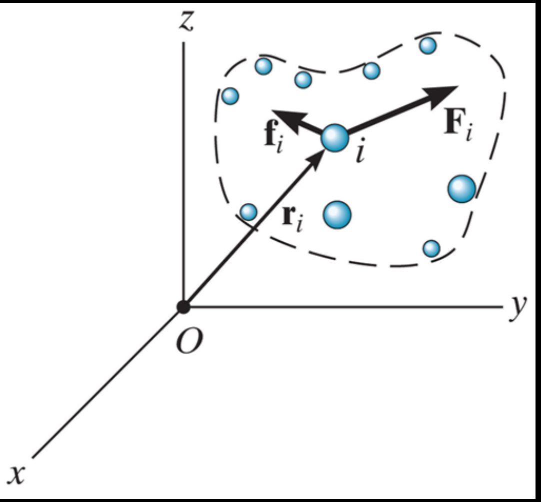
The resultant force acting on the particle is equal to the time rate of change of the particle's linear momentum. Showing the time derivative using the familiar "dot" notation results in the equation

$$\sum \boldsymbol{F} = \dot{\boldsymbol{L}} = m\dot{\boldsymbol{v}}$$

We can prove that the resultant moment acting on the particle about point O is equal to the time rate of change of the particle's angular momentum about point O or

$$\sum \boldsymbol{M}_o = \boldsymbol{r} \times \boldsymbol{F} = \dot{\boldsymbol{H}}_o$$

MOMENT AND ANGULAR MOMENTUM OF A SYSTEM OF PARTICLES



$$\sum \boldsymbol{M}_o = r \times \boldsymbol{F} = \dot{\boldsymbol{H}}_o$$

The same form of the equation can be derived for the system of particles.

The forces acting on the i-th particle of the system consist of a resultant external force $\boldsymbol{F}_i$ and a resultant internal force $\boldsymbol{f}_i$.

Then, the moments of these forces for the particles can be written as $\sum(r_i \times \boldsymbol{F}_i) + \sum(r_i \times \boldsymbol{f}_i) = \sum(\dot{\boldsymbol{H}}_i)_o$

The second term is zero since the internal forces occur in equal but opposite collinear pairs. Thus,

$$\sum \boldsymbol{M}_o = \sum(r_i \times \boldsymbol{F}_i) = \sum(\dot{\boldsymbol{H}}_i)_o$$

Angular impulse and momentum principles
(Section 15.7)

Considering the relationship between moment and time rate of change of angular momentum

$$\sum \boldsymbol{M}_o = \dot{\boldsymbol{H}}_o = d\boldsymbol{H}_o/dt$$

By integrating between the time interval t_1 to t_2

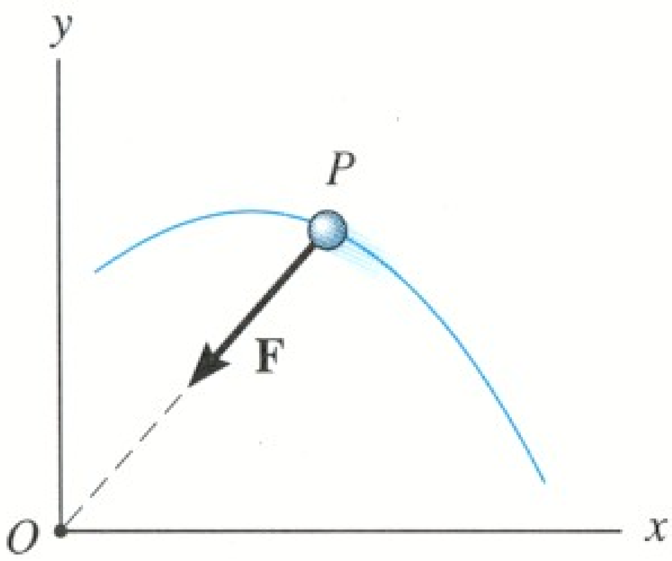
$$\sum \int_{t_1}^{t_2} \boldsymbol{M}_o \, dt = (\boldsymbol{H}_o)_2 - (\boldsymbol{H}_o)_1 \quad \text{or} \quad (\boldsymbol{H}_o)_1 + \sum \int_{t_1}^{t_2} \boldsymbol{M}_o \, dt = (\boldsymbol{H}_o)_2$$

This equation is referred to as the principle of angular impulse and momentum. The second term on the left side, $\sum\int M_o \, dt$, is called the angular impulse. In cases of 2D motion, it can be applied as a scalar equation using components about the z-axis.

Conservation of angular momentum

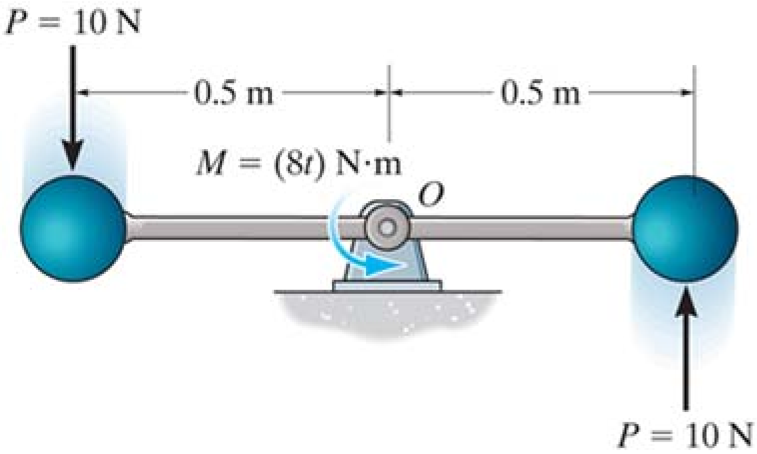
When the sum of angular impulses acting on a particle or a system of particles is zero during the time t_1 to t_2, the angular momentum is conserved. Thus,

$$(\boldsymbol{H}_O)_1 = (\boldsymbol{H}_O)_2$$



An example of this condition occurs when a particle is subjected only to a central force. In the figure, the force ***F*** is always directed towards point O. Thus, the angular impulse of ***F*** about O is always zero, and angular momentum of the particle about O is conserved.

EXAMPLE



Given: Two identical 10-kg spheres are attached to the rod, which rotates in the horizontal plane. The spheres are subjected to tangential forces of P = 10 N, and the rod is subjected to a couple moment M = (8t) N·m, where t is in seconds.

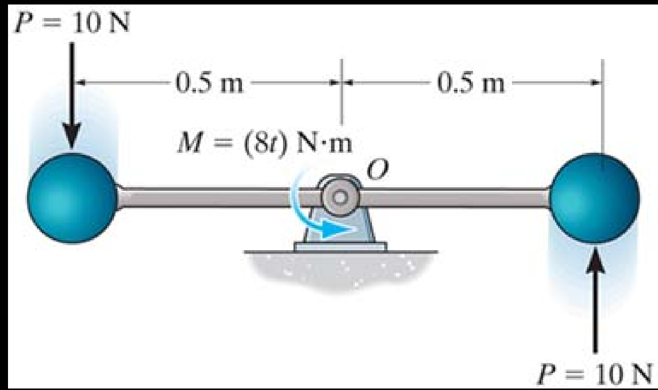
Find: The speed of the spheres at t = 4 s, if the system starts from rest.

Plan: Apply the principles of conservation of energy and conservation of angular momentum to the system.

EXAMPLE
(continued)

Solution:

Conservation of angular momentum :



$$\Sigma(\boldsymbol{H}_0)_1 + \Sigma \int_{t1}^{t2} \boldsymbol{M}_0 dt = \Sigma(\boldsymbol{H}_0)_2$$

The above equation about the axis of rotation (z-axis) through O can be written as

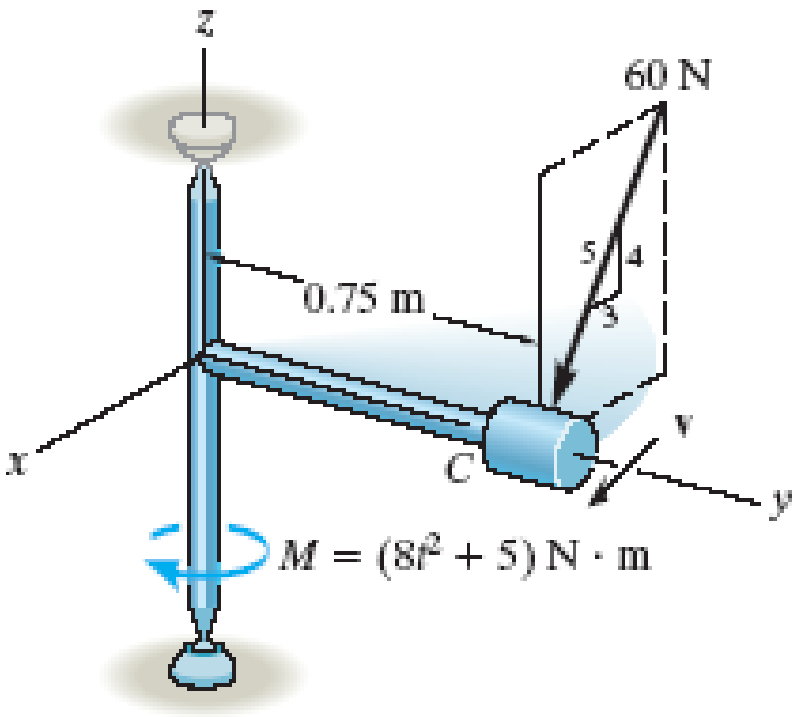
$$0 + \int_0^4 8t\,dt + \int_0^4 [2(10)(0.5)]\,dt = 2\,[10\,v\,(0.5)]$$

$$\Rightarrow 4\,(4)^2 + 2\,(5)4 = 10\,v$$

$$\Rightarrow 104 = 10\,v$$

$$v = 10.4 \text{ m/s}$$

Example



Given: A rod assembly rotates around its z-axis. The mass C is 10 kg and its initial velocity is 2 m/s. A moment and force both act as shown ($M = 8t^2 + 5$ N·m and $F = 60$ N)

Find: The velocity of the mass C after 2 seconds

Plan: Apply the principle of angular impulse and momentum about the axis of rotation (z-axis)

Example (continues)

Solution:

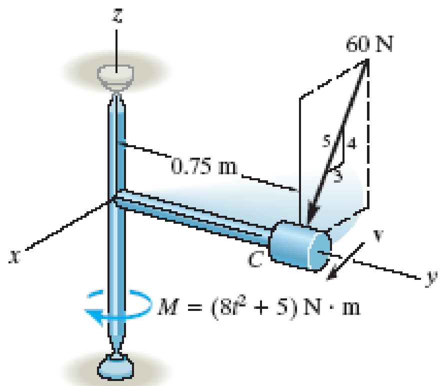
Angular momentum: $\boldsymbol{H}_Z = \boldsymbol{r} \times m\boldsymbol{v}$ reduces to a scalar equation.

$(H_Z)_1 = 0.75(10)(2) = 7.5(2)$ and $(H_Z)_2 = 0.75(10)(v_2) = 7.5v_2$

Angular impulse: $\int_{t_1}^{t_2} M\,dt + \int_{t_1}^{t_2} (\boldsymbol{r} \times \boldsymbol{F})\,dt$

$$= \int_0^2 (8t^2 + 5)\,dt + \int_0^2 (0.75)(3/5)(60)\,dt$$

$$= (8/3)t^3 + 5t + 27t \Big|_0^2 = 85.33 \text{ N·m·s}$$



Apply the principle of angular impulse and momentum.

$$7.5(2) + 85.33 = 7.5v \qquad \boxed{v = 13.38 \text{ m/s}}$$

W. Wang

Final Project

1. **Proposal for design project (GROUPS)**

The CS student finally realizes the meaning of the word "deadline".

Deadline: Proposal due This Friday Feb. 14

Homework Assignment

Chapter15-6,11, 21,42, 54,57
Chapter15- 63, 80, 88, 92,105,108

Due next Wednesday !!!

W. Wang