EARTH AND SPACE SCIENCE **Autumn 2018**

431 *PRINCIPLES OF GLACIOLOGY* 4 Credits, SLN 14855

505 *THE CRYOSPHERE* 4 Credits, SLN 14871

Lab Week 3 – Snowball Earth and Snow Growth

Question 1: Dendritic Snow Growth

In Figure 4.1 of Marshall, dendritic snowflake growth is likely to occur in supersaturated regimes. Dendritic growth will happen most rapidly when the equilibrium vapor pressure of super-cooled liquid water is greatest compared to the equilibrium vapor pressure of ice. Use the Clausius-Clapeyron equation to find the temperature at which you would expect the fastest dendritic growth of snowflakes. [You may assume that the latent heats of sublimation and evaporation are independent of temperature and just use their values at 0°C.]

Useful Information:

Latent heat of sublimation: 51.2 kJ/mol
Latent heat of evaporation: 45.2 kJ/mol
Ideal gas law constant: 8.37 J/(mol K)
Triple point temperature: 273.16 K

Clausius-Clapeyron Relation:

$$\frac{dP}{dT} = \frac{PL}{RT^2}$$

[Watch your units]

P = Pressure (Pa)
T = Temperature (K)
L = Latent heat for phase change (J/kg)
R =ideal gas constant (J/(mole K)

(a) Integrate the Clausius-Clapeyron equation from the triple point to an arbitrary equilibrium vapor pressure as a function of temperature for both ice and water.

We can separate variables and integrate:

$$\int_{P_0}^{P} \frac{dP}{P} = \frac{L}{R} \int_{T_0}^{T} T^{-2} dT$$

$$\ln\left(\frac{P}{P_0}\right) = \frac{L}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)$$

Then substituting in the vapor saturation pressure for either water or ice gives:

$P_i = P_0 \exp\left[\frac{L_i}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]$ for the vapor saturation pressure over ice

and

$P_w = P_0 \exp\left[\frac{L_w}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]$ for the vapor saturation pressure over ice

where $L_{i,w}$ represent the latent heat for a change from the condensed phase (ice or water, respectively) to the vapor phase.

(b) Using calculus, solve for the temperature at which the difference between equilibrium vapor pressure over ice and equilibrium vapor pressure of water is maximized.

$$\frac{d}{dT}(P_w - P_i) = \frac{P_i L_w}{RT^2}\exp\left[\frac{L_w}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right] - \frac{P_0 L_i}{RT^2}\exp\left[\frac{L_i}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]$$

Setting this expression equal to zero then yields:

$$\frac{P_i L_w}{RT^2}\exp\left[\frac{L_w}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right] = \frac{P_0 L_i}{RT^2}\exp\left[\frac{L_i}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]$$

and canceling and simplifying gives:

$$L_w \exp\left[\frac{L_w}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right] = L_i \exp\left[\frac{L_i}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]$$

$$\frac{L_i}{L_w} = \frac{\exp\left[\frac{L_w}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]}{\exp\left[\frac{L_i}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]}$$

and

$$\frac{L_w}{RT_0} - \frac{L_w}{RT} - \frac{L_i}{RT_0} + \frac{L_i}{RT} = \ln\left(\frac{L_i}{L_w}\right)$$

So

$$\frac{1}{T} = \frac{1}{T_0} + \frac{R}{L_i - L_w}\ln\left(\frac{L_i}{L_w}\right)$$

(c) Substitute in the values above to find the temperature at which the difference between equilibrium vapor pressure of water versus ice is maximized.

For $L_i = 51.2$ kJ/mol, $L_w = 45.2$ kJ/mol, R =8.37 J/(mol K), and $T_0 = 273.16$ K, $T = 260.8$ K $= -12.4$°C.

Question 2: Ice on a Snowball

Fourier's Law: $Q = -k\frac{dT}{dz}$	Q = Heat flux (W m^{-2})
	T = Temperature (K)
	z = Depth (m)
[Watch your units]	k = 2.1 W m^{-1} K^{-1} (thermal conductivity of ice)

Thin ice (~1 m) is common on the modern ocean; why not on Snowball Earth? To answer this question, work the following exercise.

(a) Compute the equilibrium sea-ice thickness for typical values in the winter for the ocean around Antarctica: surface air temperature -20°C, freezing temperature of seawater -2°C, heat flux from the ocean to the underside of the ice 15 W m^{-2}. Is your answer larger or smaller than the observed thickness of 1 m? Can you think of a reason for the difference?

Fourier's law gives

$$Q = -k\frac{dT}{dz}$$

Substituting in values and discretizing then gives:

$$15 = -2.1\frac{\Delta T}{\Delta z} = -2.1\frac{-18}{\Delta z}$$

so

$$\Delta z = 2.5\text{m}$$

This number is larger than the observed thickness because sea ice doesn't usually reach 2.5m thickness in a single winter, so melting occurs in summer, and because the temperature of the surface is not a constant through the winter, so full equilibration cannot occur.

(b) That heat flux from the ocean in part (a) originates from solar heating of the ocean surface water at lower latitudes; the warm water flows poleward and under the ice. But on Snowball Earth, after the ocean is covered by ice, there is no more solar heating of the tropical ocean, so over a few thousand years the ocean loses its reservoir of heat by conduction upward through the ice. Thereafter, the ocean water below the ice is at the freezing point of seawater everywhere, -2°C, and the only source of heat from the ocean to the underside of the ice is geothermal heat transmitted upward through the water. A typical value of geothermal heat for the oceanic crust is 0.08 W m^{-2}. Now compute the equilibrium thickness of ice on the ocean, with the same surface temperature as in part (a).

$$0.08 = -2.1\frac{-18}{\Delta z} \xrightarrow{yields} \Delta z = 470\text{m}$$