EARTH AND SPACE SCIENCE **Autumn 2018**
431 *PRINCIPLES OF GLACIOLOGY* 4 Credits, SLN 14855
505 *THE CRYOSPHERE* 4 Credits, SLN 14871

Lab Week 5 – Heat Flow and Temperature in Ice

Discretizing the Heat Flow Equation (or any diffusion)

We wish to model non-steady one-dimensional homogeneous heat conduction with arbitrary boundary conditions (no advection in this discretization). Heat conduction is diffusion process so can be described by the diffusion equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

where T is temperature, t is time, α is the thermal diffusivity, and x is a position variable (could be considered depth for most of the problems we did in class).

If we assume we can approximate the derivatives as finite differences, we can rewrite this equation by component, as, for the time derivative first:

$$\left.\frac{\partial T}{\partial t}\right|_{i+1/2,j} \approx \frac{T_j^{i+1} - T_j^i}{\Delta t}$$

where i is an arbitrary time index and j is an arbitrary spatial index. The spatial second derivative is given by:

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j} \approx \frac{\left.\frac{\partial T}{\partial x}\right|_{i,j+\frac{1}{2}} - \left.\frac{\partial T}{\partial x}\right|_{i,j-\frac{1}{2}}}{\Delta x} = \frac{\frac{T_{j+1}^i - T_j^i}{\Delta x} - \frac{T_j^i - T_{j-1}^i}{\Delta x}}{\Delta x} = \frac{T_{j+1}^i - 2T_j^i + T_{j-1}^i}{\Delta x^2}$$

Now equating both sides of the equation gives:

$$\frac{T_j^{i+1} - T_j^i}{\Delta t} = \alpha \frac{T_{j+1}^i - 2T_j^i + T_{j-1}^i}{\Delta x^2}$$

If we rewrite in terms of the value at the next time step, we have:

$$T_j^{i+1} = T_j^i + \frac{\alpha \Delta t}{\Delta x^2}\left(T_{j+1}^i - 2T_j^i + T_{j-1}^i\right)$$

Note that if Δx (the denominator) is too small, we are multiplying by a very large number and the equation will "blow up" or be unstable. We can define a stability criterion by stipulating that this number must be less than 1 for all terms when we expand this equation on the right hand side. That gives the stability criterion $\sqrt{2\alpha\Delta t \leq} \Delta x$. If you implement this simple scheme in a nested loop (in time and space), you can solve the heat equation (or any diffusion equation) for an arbitrary boundary condition without making guess or calculating challenging integrals. This basic numerical approach using finite differences is rather powerful and can be used to address problems of heat flow, hillslope evolution, elastic flexure of tectonic plates and much more. It should be a skill in any earth scientist's toolkit.