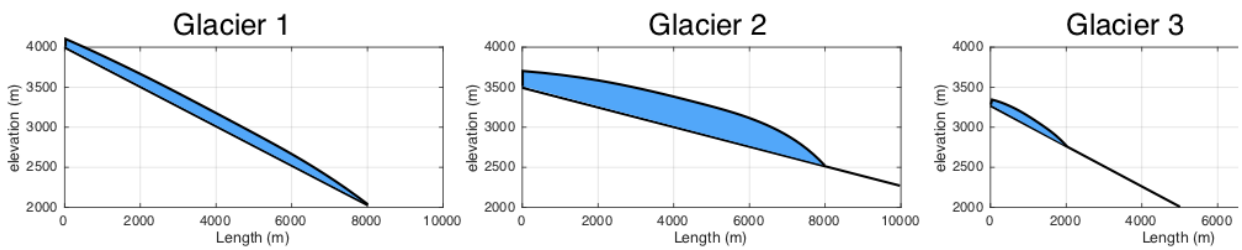
EARTH AND SPACE SCIENCE **Autumn 2018**
431 *PRINCIPLES OF GLACIOLOGY* 4 Credits, SLN 14855
505 *THE CRYOSPHERE* 4 Credits, SLN 14871

Lab Week 6 – Glacier Variations (Solutions)

I. Glacier Equilibrium Response to a Change in Climate

Consider three simple glacier geometries. Assume they reside in the same climate, all with an Equilibrium Line Altitude (ELA) of 3000m, and mass balance gradient $\frac{db}{dz}$ of 8 (m/yr)/km. All have constant widths.



- Glacier 1 starts at an elevation of Z = 4000m and terminates at 2000m. Its steady state length, L_0, is 8000m. Its characteristic thickness is 80m.
- Glacier 2 starts at an elevation of Z = 3500m and terminates at 2500m. Its steady state length, L_0, is 8000m. Its characteristic thickness is 160m.
- Glacier 3 starts at an elevation of Z = 3250m and terminates at 2750m. Its steady state length, L_0, is 2000m. Its characteristic thickness is 75m.

1) Find the response times and equilibrium sensitivities (dL/db) for each glacier. See if you can arrive at a compact symbolic expression for equilibrium sensitivity (valid for these simple geometries). You'll need it later.

We will start to solve this problem by determining the mathematical expressions needed.

For each glacier we know where the equilibrium line is (3000m elevation). Thus we can use the given vertical gradient in mass balance to calculate the glacier's terminus balance relative to the balance at the equilibrium line, which by definition is 0, or:

$$b_t = \frac{db}{dz}\Delta z$$

In class on Monday (you will also derive this in homework #5), we determined the characteristics response time:
$\tau = -\frac{H_0}{b_t}$ where H_0 is the characteristic thickness and b_t is the specific mass balance at the terminus at the glacier terminus.
The glaciers are initially in balance (no net change in mass or length) and have constant widths (so we can neglect the width term), so any change in balance will results in a change in length of the glacier. We can determine the change in length through mass flux conservation. If db is the spatial average of the change in specific mass balance

($db = \int_0^{L_0} b_1(x)dx$, where $b_1(x)$ is your perturbed mass balance), then the change in mass flux at the glacier's original terminus is $db \times L_0$. For a positive change db, this flux must be ablated away over the length the glacier advances, thus:

$$db \times L_0 = dL \times a_0$$

where dL is the change in length and a_0 is the ablation rate (positive if the mass is being lost, i.e., the glacier's surface is lowering). Rewriting this equation for the change in length for a given change in mass balance (the glacier's equilibrium sensitivity) gives:

$$\frac{dL}{db} = \frac{L_0}{a_0}$$

Let's think about what this means for one minute (we will need this for part 2). For glacier 1, we can calculate this as:

$$\frac{dL}{db} = \frac{L_0}{a_0} = \frac{8000}{8} = 1000\,\frac{\text{m}}{\text{m/yr}}$$

Thus, if the mass balance changes by a spatial average of 0.5 m/yr (ice equivalent), the glacier would advance by:

$$\frac{dL}{db} = \frac{dL}{db}db = 1000\,\frac{\text{m}}{\text{m/yr}} \times 0.5\,\frac{\text{m}}{\text{yr}} = 500\text{ m}$$

Conversely, if the if the mass balance changes by a spatial average of -0.5 m/yr, the glacier would retreat by 500m. Note that from the relationships above describe different aspects of the glacier's behavior. The characteristic time is an e-folding time describing how long the glacier takes to reach its new equilibrium length. The dL is the glacier's change in length, but that relationship does not contain information on how long that change takes to occur.

Now we are in the position to calculate these quantities for each glacier.

Glacier 1:

$$b_t = \frac{db}{dz}\Delta z = 8\,\frac{\text{m/yr}}{\text{km}} \times -1\text{ km} = -8\text{ m/yr}$$

$$\tau = -\frac{H_0}{b_t} = -\frac{80\text{m}}{-8\,\frac{\text{m}}{\text{yr}}} = 10\text{ yrs}$$

$$\frac{dL}{db} = \frac{L_0}{a_0} = \frac{8000\text{m}}{8\text{ m/yr}} = 1000\,\frac{\text{m}}{\text{m/yr}} = 1\,\frac{\text{km}}{\text{m/yr}}$$

Glacier 2:

$$b_t = \frac{db}{dz}\Delta z = 8\,\frac{\text{m/yr}}{\text{km}} \times -0.5\text{ km} = -4\text{ m/yr}$$

$$\tau = -\frac{H_0}{b_t} = -\frac{160\text{m}}{-4\,\frac{\text{m}}{\text{yr}}} = 40\text{ yrs}$$

$$\frac{dL}{db} = \frac{L_0}{a_0} = \frac{8000\text{m}}{4\text{ m/yr}} = 2000\,\frac{\text{m}}{\text{m/yr}} = 2\,\frac{\text{km}}{\text{m/yr}}$$

Glacier 3:

$$b_t = \frac{db}{dz}\Delta z = 8\,\frac{\text{m/yr}}{\text{km}} \times -0.25\text{ km} = -2\text{ m/yr}$$

$$\tau = -\frac{H_0}{b_t} = -\frac{75\text{m}}{-2\,\frac{\text{m}}{\text{yr}}} = 37.5\text{ yrs}$$

$$\frac{dL}{db} = \frac{L_0}{a_0} = \frac{2000\text{m}}{2\text{ m/yr}} = 1000\,\frac{\text{m}}{\text{m/yr}} = 1\,\frac{\text{km}}{\text{m/yr}}$$

2) You are doing fieldwork on glacier #2 and, on your lunch break, ski over a small pass and find another glacier on the same mountain. You're at the head of the glacier, which your GPS tells you is at 3200m. You can see that it has the same slope, and assume being close by and same aspect, it has the same mass balance gradient and ELA. But, it's a bit cloudy down below so you can't see the terminus.

a. Estimate the new glacier's length and terminus elevation from what you know about glacier #2.

This glacier has the same ELA as glacier #2, which is 3000 m elevation. Since it is in steady state, the ELA must be halfway in elevation between the head of the glacier and its terminus, or is at 2800 m. Once we know it has the same slope as glacier #2, we can calculate its length too. Glacier #2 has a length of 8000 m and an elevation range of 1000 m, so its slope is 1/8. So, for this new glacier, we have from the definition of slope:
$\frac{\Delta z}{L_0} = \frac{400\text{m}}{L_0} = \frac{1}{8} \xrightarrow{yields} L_0 = 8 \times 400 \text{ m} = 3200 \text{ m}.$

b. You don't know the thickness of this new glacier, but using what you know about glacier #2, estimate this glacier's response time and sensitivity.

We can estimate the glacier's response time and sensitivity using the same process as for number 1.

New Glacier:

$$b_t = \frac{db}{dz}\Delta z = 8\frac{\text{m/yr}}{\text{km}} \times -0.2 \text{ km} = -1.6 \text{ m/yr}$$

$$\tau = -\frac{H_0}{b_t} = -\frac{160\text{m}}{-1.6\frac{\text{m}}{\text{yr}}} = 100 \text{ yrs}$$

$$\frac{dL}{db} = \frac{L_0}{a_0} = \frac{3200\text{m}}{1.6 \text{ m/yr}} = 2000\frac{\text{m}}{\text{m/yr}} = 2\frac{\text{km}}{\text{m/yr}}$$

where we have assumed the glacier is the same thickness as glacier #2. This is probably not the best assumption, even if it is the simplest. A much better assumption might be to scale the glacier's thickness by the relative area of the glacier, as the area determines the total flux that much be conserved (accumulation above the ELA, ablation below the ELA). If we make this assumption, this new glacier is only 64m thick. And the response time and sensitivity would be:

$$\tau = -\frac{H_0}{b_t} = -\frac{64\text{m}}{-1.6\frac{\text{m}}{\text{yr}}} = 40 \text{ yrs}$$

$$\frac{dL}{db} = \frac{L_0}{a_0} = \frac{3200\text{m}}{1.6 \text{ m/yr}} = 2000\frac{\text{m}}{\text{m/yr}} = 2\frac{\text{km}}{\text{m/yr}}$$

c. Given what you know about ice dynamics, explain why you think this estimate is likely an over or under estimate. You don't have to solve for the thickness, but explain the physical basis for your answer.
**hint: think about last homework on kinematics and dynamics*

The assumption of the same thickness as glacier #2 is clearly a vast overestimate, as this glacier has a much smaller accumulation area and the same surface slopes, so smaller balance fluxes would be needed to evacuate the accumulated mass. A better assumption would be to scale by area as we just did, and then it's easy to see that this glacier is most likely to respond to climate change in an identical way to glacier #2, despite its different size and terminus ablation rate (the geometry would compensate for these differences to reach a steady state)!!

II. Glacier Transient Response to a Change in Climate

The simplest model for transient response is exponential (as discussed in reading and homework). The transient length solution for trend in mass balance ($\dot{b} \equiv \frac{db}{dt}$) starting at t = 0 is:

$$L'(t) = \frac{L_0}{a_0}\dot{b}\left[t - \tau\left(1 - e^{-t/\tau}\right)\right]$$

where a_0 is the ablation rate at the terminus and τ is the glacier's response time.

1) If glaciers 1—3 are subject to the same trend of -0.005 (m/yr)/yr, after 100 years how far out of equilibrium are they? (solve for the length difference between transient and equilibrium responses).

   This first problem is mainly "plug and chug". For each glacier, we have (using the values from part I:

   Glacier 1:

   - The total change in length (total length change in response to this balance trend is:

$$dL = \frac{dL}{db} \times db = 1000\,\frac{\text{m}}{\frac{\text{m}}{\text{yr}}} \times -0.005\,\frac{\frac{\text{m}}{\text{yr}}}{\text{yr}} \times 100\text{ yrs} = -500\text{ m}$$

   - The change in length after 100 years is

$$L'(100\text{ yrs}) = \frac{8000\text{ m}}{8\,\frac{\text{m}}{\text{yr}}} \times -0.005\,\frac{\frac{\text{m}}{\text{yr}}}{\text{yr}} \times \left[100\text{ yrs} - 10\text{ yrs}\left(1 - e^{-\frac{100\text{ yrs}}{10\text{ yrs}}}\right)\right] = -450\text{ m}$$

   so this glacier is committed to 50 m additional retreat 100 years after the trend in mass balance started before a new equilibrium (steady state) is reached.

   Glacier 2:

   - The total change in length (total length change in response to this balance trend is:

$$dL = \frac{dL}{db} \times db = 2000\,\frac{\text{m}}{\frac{\text{m}}{\text{yr}}} \times -0.005\,\frac{\frac{\text{m}}{\text{yr}}}{\text{yr}} \times 100\text{ yrs} = -1000\text{ m}$$

   - The change in length after 100 years is

$$L'(100\text{ yrs}) = \frac{8000\text{ m}}{4\,\frac{\text{m}}{\text{yr}}} \times -0.005\,\frac{\frac{\text{m}}{\text{yr}}}{\text{yr}} \times \left[100\text{ yrs} - 40\text{ yrs}\left(1 - e^{-\frac{100\text{ yrs}}{40\text{ yrs}}}\right)\right] = -630\text{ m}$$

   so this glacier is committed to 370 m additional retreat 100 years after the trend in mass balance started before a new equilibrium (steady state) is reached.

Glacier 3:

- The total change in length (total length change in response to this balance trend is:

$$dL = \frac{dL}{db} \times db = 1000 \frac{\text{m}}{\frac{\text{m}}{\text{yr}}} \times -0.005 \frac{\frac{\text{m}}{\text{yr}}}{\text{yr}} \times 100 \text{ yrs} = -500 \text{ m}$$

- The change in length after 100 years is

$$L'(100 \text{ yrs}) = \frac{2000 \text{ m}}{2 \frac{\text{m}}{\text{yr}}} \times -0.005 \frac{\frac{\text{m}}{\text{yr}}}{\text{yr}} \times \left[100 \text{ yrs } - 37.5 \text{ yrs}\left(1 - e^{-\frac{100 \text{ yrs}}{37.5 \text{ yrs}}}\right)\right] = -325 \text{ m}$$

so this glacier is committed to 175 m additional retreat 100 years after the trend in mass balance started before a new equilibrium (steady state) is reached.

2) Is the difference due to their different response times or different sensitivity? To find out, modify your expression for equilibrium sensitivity from part I so that it expresses the equilibrium response to a trend as a function of time. Then, divide the transient response (above) by the equilibrium response to get an expression for *fractional adjustment.*

To find the fractional length adjustment, we simplify divide the transient length solution by the total length adjustment (or the total length change), which is you note the calculation above is symbolically given by:

$$dL = \frac{dL}{db} \times db = \frac{L_0}{a_0} \times \dot{b} \times t = \frac{L_0}{a_0} \dot{b} t$$

Dividing the transient length solution by this term gives:

$$\frac{L'(t)}{dL} = \frac{\frac{L_0}{a_0} \dot{b}\left[t - \tau\left(1 - e^{-t/\tau}\right)\right]}{\frac{L_0}{a_0} \dot{b} t} = 1 - \frac{\tau}{t}\left(1 - e^{-t/\tau}\right)$$

as the fractional adjustment of a glacier at any given time after the start of a trend.

3) What does it depend on? What is it for each glacier, after 100 years?

This expression depends only on the time since the trend in mass balance started and the characteristic response time of the glacier (so there is an implicit dependence on the characteristic thickness and terminus mass balance). For each glacier, we have:

Glacier 1:

$$\frac{L'(t)}{dL} = 1 - \frac{\tau}{t}\left(1 - e^{-\frac{t}{\tau}}\right) = 1 - \frac{10 \text{ yrs}}{100 \text{ yrs}}\left(1 - e^{-\frac{100 \text{ yrs}}{10 \text{ yrs}}}\right) = 0.9$$

Glacier 2:

$$\frac{L'(t)}{dL} = 1 - \frac{\tau}{t}\left(1 - e^{-\frac{t}{\tau}}\right) = 1 - \frac{40 \text{ yrs}}{100 \text{ yrs}}\left(1 - e^{-\frac{100 \text{ yrs}}{40 \text{ yrs}}}\right) = 0.63$$

Glacier 3:

$$\frac{L'(t)}{dL} = 1 - \frac{\tau}{t}\left(1 - e^{-\frac{t}{\tau}}\right) = 1 - \frac{37.5 \text{ yrs}}{100 \text{ yrs}}\left(1 - e^{-\frac{100 \text{ yrs}}{37.5 \text{ yrs}}}\right) = 0.65$$

Note that these values are the same as those you would get from using your solution from problem 2 above dividing the change in length after 100 years by the total length change.

4) Now, consider two more glaciers. We don't know their mass balance gradients or sensitivities. But we do know their response times (10 and 50 years), and have observed both retreat 800m in the last 100 years. If we again assume a mass balance trend started 100 years ago, estimate how far each of these glaciers would retreat if the mass balance trend stopped today.

We can calculate their fractional retreats.

For the first glacier:

$$\frac{L'(t)}{dL} = 1 - \frac{\tau}{t}\left(1 - e^{-\frac{t}{\tau}}\right) = 1 - \frac{10 \text{ yrs}}{100 \text{ yrs}}\left(1 - e^{-\frac{100 \text{ yrs}}{10 \text{ yrs}}}\right) = 0.9$$

and then

$$L_{eq} = dL = \frac{L'(t)}{0.9} = \frac{800\text{m}}{0.9} = 890 \text{ m}$$

so an additional 90 m of retreat would occur if the forcing stopped today.

For the second glacier:

$$\frac{L'(t)}{dL} = 1 - \frac{\tau}{t}\left(1 - e^{-\frac{t}{\tau}}\right) = 1 - \frac{50 \text{ yrs}}{100 \text{ yrs}}\left(1 - e^{-\frac{100 \text{ yrs}}{50 \text{ yrs}}}\right) = 0.57$$

and then

$$L_{eq} = dL = \frac{L'(t)}{0.57} = \frac{800\text{m}}{0.57} = 1400 \text{ m}$$

so an additional 600 m of retreat would occur if the forcing stopped today.

These glaciers have characteristic times scales that are similar to the local glaciers in the Pacific Northwest. The first glacier (with the shorter characteristic timescale) might represent a glacier on a big volcano (like Mt. Rainer or Mt. Baker), and the second glacier (with the longer characteristic timescale) might represent a valley/cirque glacier in the North Cascades. So the glaciers on the Cascade volcanoes are likely much more closely adjusted to their climate due to their ability to respond rapidly to a climate change. The glaciers in the North Cascades are likely to be more out of equilibrium and have more committed retreat.

Finally, please note that this is the simplest model, which has some known issues; namely, the model assumes the terminus starts responding immediately (exponential decay) and that mass balance rate at the terminus is fixed in time, when it would really change as the glacier's length changes. More complex models show a spin-up phase of retreat, that is due to the time delay between the forcing and the thickness changes that are necessary to drive flux changes and advance/retreat. This would slow the response of the glacier, increasing the lag between the glacier's current length and its new equilibrium length, and thus also increasing disequilibrium.