Determining a Vertical Along-Flow Velocity Profile in an Ice-Sheet Flowline model

We start by considering the driving stress for a simple glacier flow geometry, as drawn below (in Figure 1a), of a glacier flowing down an inclined slope. This is the same situation and geometry that was used in class, but the z-coordinate now increases upwards towards the glacier surface.

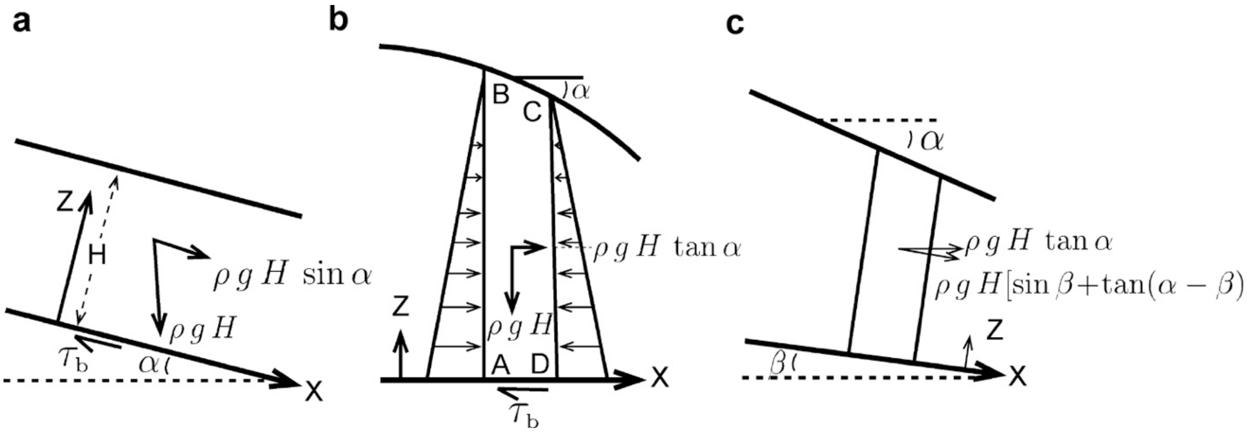

Figure 1: Different geometries to consider for calculating glacier or ice sheet shear stress on a flowline (in an xz-plane).

In what follows, we follow the case for Figure 1a. Here the main difference from what we did in class is that z increases from the bed to the glacier's surface. In this case, the bed-parallel shear stress ($\sigma_{xz}(z)$), is determined from a force-diagram analogous to a block on an incline plane (see lecture notes from 17 Oct 2018), and is given by:

$$\sigma_{xz}(z) = \rho g(h - z)\sin\theta$$

here z is height above the glacier's bed, $g = 9.81$ m/s^2, ρ is the density of ice, θ is the surface slope, and h is the glacier's total thickness. We can determine the strain rate by applying the constitutive relation, which relates deformation rate to the stress applied. For ice that is given by Glen's flow law, which for a single applied stress can be written as

$$\dot{\epsilon}_{xz} = A{\sigma_{xz}}^n$$

where A is a rheological constant determining how soft ice is as a function of temperature and n is an exponent that may be different for different situations (single crystal vs. polycrystalline ice, strain history, etc.), but empirically is should to be $n \approx 3$ for glacier and ice sheet flow. Applying the definition of shear stress gives:

$$\frac{1}{2}\frac{\partial u}{\partial z} = A(\rho g(h - z)\sin\theta)^n$$

Then we can integrate in the vertical direction from $z = 0$ at the glacier's bed to an arbitrary height z above the glacier's bed to find the speed in the along-flow direction (x-direction as shown above), $u(z)$. However, before we start blindly integrating, let's consider the form of shear stress equation and the equations above to see if we can come up with a more general expression that would hold for other geometries (like that shown in Figures 1b and 1c). We frequently call $\sigma_{xz}(z)$ the driving stress because it is a force per unit area (the definition of a stress and, in this case, it is gravity (the weight of the ice above) that loads the ice and causes it to deform, or drives ice sheet motion. However, in reality the force in question acts on the entire column, not on a particular surface, so $\sigma_{xz}(z)$ is not a true stress. Furthermore, for equilibrium (non-accelerating considerations which are valid for glaciers), resisting forces must balance this driving stress. So, we really must integrate the stresses over the entire column to properly consider this a drive stress. It is not particularly difficult to derive more generalized expression for driving stress. You can do so by integrating the stress between two flux gates, and adding a term for a bed parallel shearing force. In Figure 1a, there is only a bed-parallel shearing force which adds to the pressure gradient, and, in Figure 1b, there is only a normal stress difference between the two gates with no bed-parallel shearing force. Combining the vertical integrated stresses on these two scenarios acting at once (geometry in Figure 1c), results in a general expression for the driving stress. For small surface slopes (using a small angle approximation) this is:

$$\tau_d \approx \rho g h \theta$$

For the situation where there are no side shears to balance this driving stress equals the basal shear stress that resists flow, $\tau_d = \tau_b$. This is usually pretty close to true in the middle of a mountain glacier. However, there may be quite low basal shear stresses as there are large stresses due to velocity contrasts across for example ice stream shear margins. Regardless of the situation, for almost all cases

$$\tau_b = f' \tau_d$$

where f' is on the order of 1. Note that this formula does not depend on the bed shape, but only on ice thickness and surface slope.

Now if we look at the form of these equations, particularly for shear stress in xz-plane, which is,

$$\sigma_{xz}(z) = \rho g (h - z) \sin\theta$$

we note that we can rewrite this as:

$$\sigma_{xz}(z) = \rho g h \sin\theta - \rho g z \sin\theta = \tau_b \left[1 - \frac{z}{h}\right]$$

where we no longer need to know the exact form of the basal shear stress (stress at the bed of the glacier, just that it decreases linearly with increasing height. This plotted below.

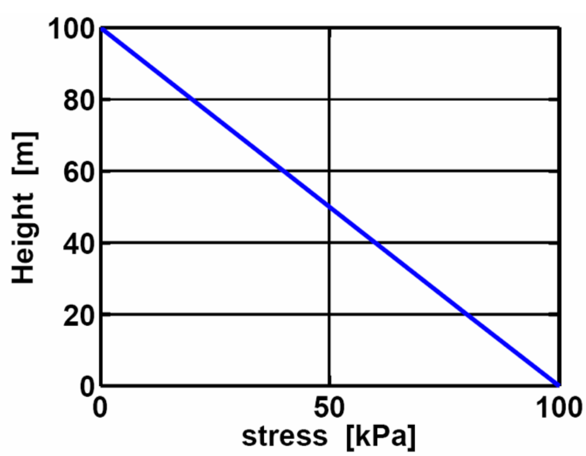

Figure 2: Decrease in shear stress in the xz-plane as a function of increasing height above the base of the glacier. Note that if you calculated this with z as depth, instead of height, it would be 0 near the surface and increase to its maximum value at the glacier's bed. This is physically identical to the situation we describe here, but you need to keep track of your coordinate systems when applying these relationships.

If we now, substitute this expression into those we had previously for strain rate using the constitutive relation for ice (Glen's flow law $\dot{\epsilon}_{xz} = A{\sigma_{xz}}^n$), we have:

$$\frac{\partial u}{\partial z} = 2A\left(\tau_b\left[1-\frac{z}{h}\right]\right)^n$$

Note that in this expression τ_b is a constant (regardless of its precise form) that does not depend on z.

Before we move on, it is useful to consider the what the strain rate is. It's the deformation rate, and when we plot the above expression (Figure 3)

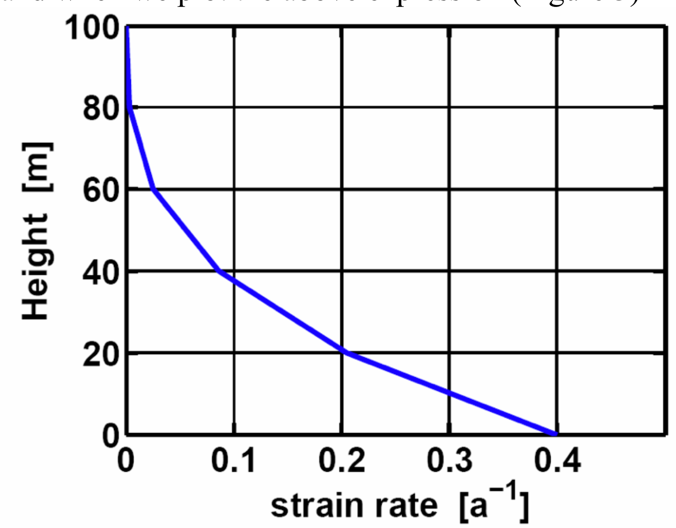

Figure 3: Strain rate as a function of height above the glacier bed. The deformation rate is greatest near the glacier's bed and lowest near the surface. This makes sense because the ice must deform due to the weight of the ice above (gravity). If there is less weight above, there is less deformation, as shown by the plot below. It is somewhat similar to Figure 2, but with a cubic rather than linear dependence, as indicated by the equations.

To determine the along-flow vertical velocity profile $u(z)$, we must integrate the deformation that has occurred of the under an arbitrary height, i.e., the velocity at any point in the ice column, is the accumulated deformation of the ice below that point.

So to find $u(z)$, we need to integrate this expression:

$$u(z) = \int_0^z 2A\left(\tau_b\left[1-\frac{z}{h}\right]\right)^n dz = 2A{\tau_b}^n\int_0^z \left[1-\frac{z}{h}\right]^n dz$$

This integral is slightly more difficult than the one we did in class but is not too bad if we make a change in variables (and here I assign m as a dummy variable that we are not using for anything else):

$$m = 1-\frac{z}{h}$$

$$dm = -\frac{dz}{h}$$

So, we can rewrite this integral as

$$u(z) = 2A{\tau_b}^{3n}\int_1^{1-\frac{z}{h}} -hm^n dm = \frac{2A{\tau_b}^n h}{n+1}\left[-m^{n+1}\right]_1^{1-\frac{z}{h}} = \frac{2A{\tau_b}^n h}{n+1}\left[-\left[1-\frac{z}{h}\right]^{n+1} - -1^{n+1}\right]$$

$$= \frac{2A{\tau_b}^n h}{n+1}\left[1-\left[1-\frac{z}{h}\right]^{n+1}\right]$$

Now if we add in a term for the basal sliding velocity:

$$u(z) = u_b + \frac{2A{\tau_b}^n h}{n+1}\left[1-\left[1-\frac{z}{h}\right]^{n+1}\right]$$

Plotting this expression gives:

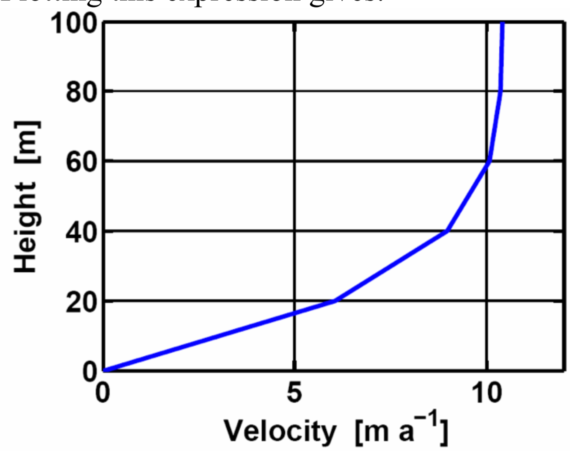

Figure 4: Speed as a function of height above the glacier bed. Note that speed is greatest at the surface and lowest at depth. Also, not that the vertical gradient in speed is highest at near the bed and decreases near the surface. This also makes sense as deformation is highest at the bed, and decreases near the surface.

Now, let's compare this to the less general expression from class at the surface, so for this coordinate system at the surface $z = h$:

$$u(h) = \frac{2A{\tau_b}^n h}{n+1}\left[1-\left[1-\frac{h}{h}\right]^{n+1}\right] = \frac{2A{\tau_b}^n h}{n+1}$$

and if we assume $\tau_b = \tau_d = \rho g h \sin\theta$ and $n = 3$ with no sliding at the glacier bed, we have

$$u_{surf} = \frac{2A{\tau_b}^n h}{n+1} = \frac{A}{2} h \tau_b^3$$

which is identical to the formula we presented in class.

Applications:

To summarize, for this geometry and frame of reference, we can completely determine the speed profile using three relationships:

1) Shear-stress in the xz-plane as a function of depth: $\sigma_{xz}(z) = \rho g(h-z)\sin\theta$
2) The u (along-flow velocity) derivative as a function of depth: $\frac{\partial u}{\partial z} = 2A(\rho g(h - z)\sin\theta)^n$.
3) The along-flow velocity (u) as a function of depth: $u(z) = u_b + \frac{2A{\tau_b}^n h}{n+1}\left[1-\left[1-\frac{z}{h}\right]^{n+1}\right]$

Problem 1: Calculations via a Finite-Difference Approach

It sometimes is useful to consider manipulating these equations in ways that don't involve integration. For example, you wish to determine the velocity difference for an object that is one meter tall at a depth of 90 meters for a glacier with a surface slope of 5° and a total ice thickness of 100 m.

Applying the equations above gives (with $A = 2 \times 10^{-16}$ $Pa^{-3}yr^{-1}$ and $n = 3$):

$$\sigma_{xz}(10) = \rho g(h - z) \sin\theta = 917 \times 9.81 \times 90 \times \sin 5° = 70 \text{ kPa}$$

$$\frac{\partial u}{\partial z} = 2A(\rho g(h - z) \sin\theta)^n = 2 \times 2 \times 10^{-16} \times (70000)^3 = 0.14 \text{ 1/yr}$$

So, for an object that is one meter tall and is at 90 m below the surface, it's top is moving

$u = \frac{du}{dz} \times dz = 0.14 \times dz = 0.14$ m/yr faster than its base due to solely the difference in ice deformation rates from the top to bottom of the object.

Problem 2: Finite Differences from Plots vs. Analytic Solution

Analytic and numerical approaches should give the same answer (assuming sufficient sampling, i.e., sufficiently small finite differences). To test this, we consider the speed 10 m above the bed for a glacier that is 100 m thick with a surface slope of 6.3°, with the basal shear stress form given above
(i.e., $\tau_b = \rho g h \sin\theta$).

$$u(10) = \frac{2A(\rho g h \sin\theta)^n h}{n+1}\left[1 - \left[1 - \frac{z}{h}\right]^{n+1}\right]$$
$$= \frac{2 \times 10^{-16} \times (917 \times 9.81 \times 100 \times \sin 6.3°)^3 \times 100}{2}[1 - 0.9^4] = 3.3 \text{ m/yr}$$

Now let's take a numerical approach (centered at $z = 5$):

$$\frac{\partial u}{\partial z} = 2A(\rho g(h - z) \sin\theta)^n = 2 \times 2 \times 10^{-16} \times (917 \times 9.81 \times 95 \times \sin 6.3°)^3 = 0.33 \text{ 1/yr}$$

Then:

$$u = \frac{du}{dz} \times dz = 0.33 \times 10 = 3.3 \text{ m/yr}$$

which is somewhat reassuring. Finally, let's do this graphically, referring back to Figure 3 and seeing on the plot that $\frac{\partial u}{\partial z} \approx 0.35$ 1/yr between 0 and 10 m height above the glacier bed, we have

$$u = \frac{du}{dz} \times dz = 0.35 \times 10 = 3.5 \text{ m/yr}$$

Thus all three methods of making this calculation (analytically, numerically from finite differences, and graphically from a plot) yield the same results. Although you don't need to integrate or derive formulas on the exam, you should be able to move between these formulas as demonstrated in the above two problems.